MATH 122

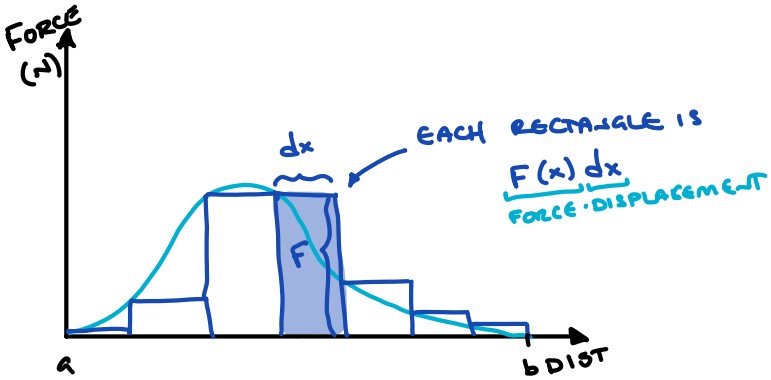
WORK

GOALS

- [ ] USE INTEGRATION TO COMPUTE WORK WITH A VARIABLE FORCE
- [ ] SET UP AND EVALUATE INTEGRALS TO COMPUTE WORK FOR THREE SCENARIOS:
  - [ ] SPRINGS
  - [ ] LIFTING
  - [ ] PUMPING.

$$\text{WORK} = \text{FORCE} \cdot \text{DISPLACEMENT}$$

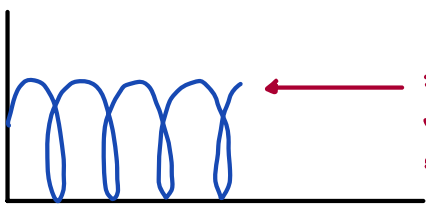
WHAT IF THIS ISN'T CONSTANT?



$$W = \int_a^b \underbrace{F(x)\,dx}_{N \cdot m}$$

← FORCE UNITS (J)

SPRINGS:



← FORCE DEPENDS ON HOW FAR WE PUSH/PULL THE SPRING FROM NATURAL LENGTH

Hookes Law: $F(x) = kx$

K: SPRING CONST.

x: DISTANCE FROM NATURAL LENGTH

x = 0

2. A force of 20 lb is required to hold a spring stretched 5 in beyond its natural length. How much work is done in stretching the spring from its natural length to 7 in beyond its natural length?

$$F(x) = kx$$

$$20 = k(5)$$

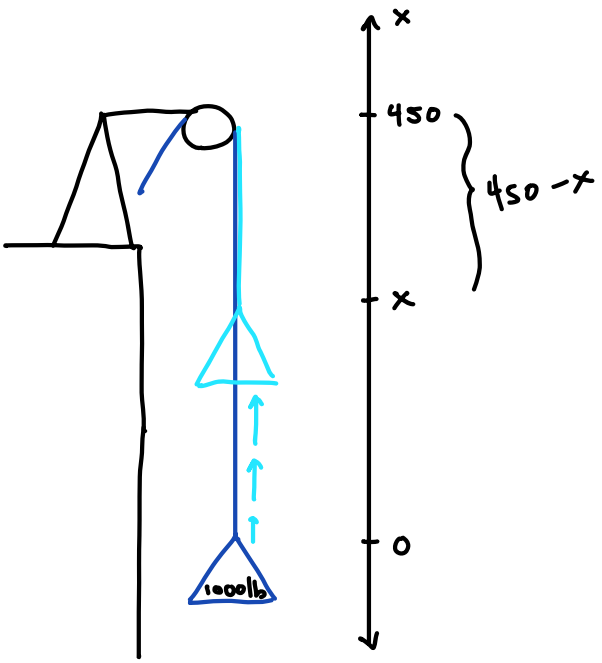
$$\boxed{k = 4}$$

$$W = \int_0^7 4x\,dx = 2x^2 \Big|_0^7 = 98 \text{ in}\cdot\text{lb}$$

$$= \frac{98}{12} \text{ ft}\cdot\text{lb}$$

LIFTING:

5. A cable that weighs 3 lb/ft is used to lift 1,000 lb of iron ore up a mineshapft which is 450 ft deep. Find the work done.

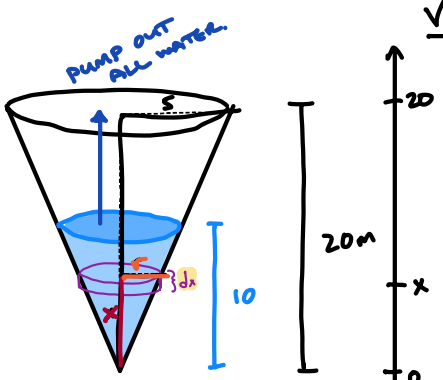


- IRON: 1000 lb
- CABLE: $3(450 - x)$ lb (3 lb/ft, $(450-x)$ ft)

* MAKE SURE UNITS MAKE SENSE *

$$W = \int_0^{450} \underbrace{(1000 + 3(450 - x))}_{\text{lb}}\,\underbrace{dx}_{\text{ft}}$$

PUMPING:



VOLUME OF "SLICE":

$\pi r^2 h$ ← dx: THIN SLICE

$\pi r^2 dx$

$\pi \left(\frac{1}{4}x\right)^2 dx$

$\frac{\pi}{16} x^2 dx$

SIMILAR TRIANGLES

$\frac{r}{5} = \frac{x}{20}$

$r = \frac{1}{4}x$

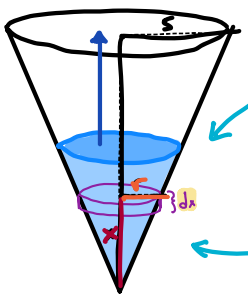
OR

$\frac{r}{x} = \frac{5}{20}$

WEIGHT OF SLICE: DENSITY = 9800

$9800 \frac{\pi}{16} x^2 dx$ ← dx IS PART OF FORCE NOT DISPLACEMENT

DISPLACEMENT: $20 - x$



$$W = \int_0^{10} 9800 \cdot \frac{\pi}{16} x^2 (20 - x)\, dx$$

$N/m^3 \cdot m^2 \cdot m \cdot m = N \cdot m$

BOUNDS REPRESENT WHERE WATER IS.

4. If the work required to stretch a spring 1 foot beyond its natural length is 15 ft·lb, how much work is needed to stretch it 8 inches beyond its natural length? (Hint: 1 foot = 12 inches)

* $\int_0^1 kx\, dx = 15$

$\frac{1}{2}kx^2 \Big|_0^1 = 15$

$\frac{1}{2}k = 15$

$k = 30$

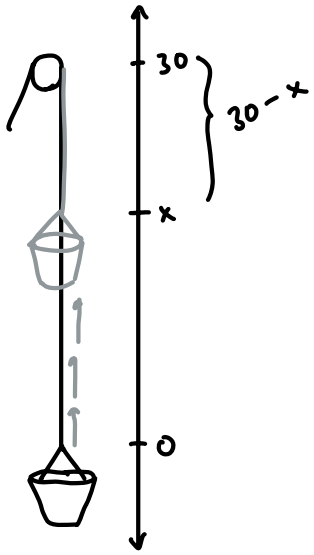
$W = \int_0^{8/12} 30x\, dx$

$= 15x^2 \Big|_0^{2/3}$

$= 15 \cdot \frac{4}{9}$

$= \boxed{\frac{20}{3}\ ft \cdot lb}$

6. A 5 lb bucket containing 10 lb of water is hanging at the end of a 30 ft rope which weighs $\frac{1}{2}$ lb/ft. The other end of the rope is attached to a pulley. Assume that the rope is wound onto the pulley at a rate of 3 ft/s causing the bucket to be lifted. Find the work done in winding the rope onto the pulley.



- BUCKET: 5lb
- WATER: 10lb
- ROPE: $\frac{1}{2}(30-x)$ lb

lb/ft · ft ← lb ✓

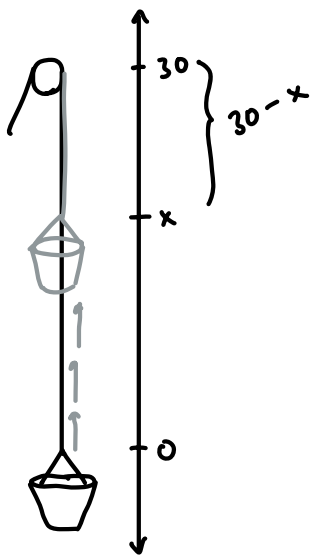
$$W = \int_0^{30} \left[5 + 10 + \frac{1}{2}(30-x)\right] dx$$

lb ft ← ft·lb ✓

6. A 5 lb bucket containing 10 lb of water is hanging at the end of a 30 ft rope which weighs $\frac{1}{2}$ lb/ft. The other end of the rope is attached to a pulley. Assume that the rope is wound onto the pulley at a rate of 3 ft/s causing the bucket to be lifted. Find the work done in winding the rope onto the pulley.

7. Repeat problem 6 adding the assumption that water leaks out of the bucket at a rate of $\frac{1}{4}$ lb/s.

$\frac{1}{3}\frac{s}{ft} \cdot \frac{1}{4} \cdot \frac{lb}{s} = \frac{1}{12}$ lb/ft ← RATE OF LEAK

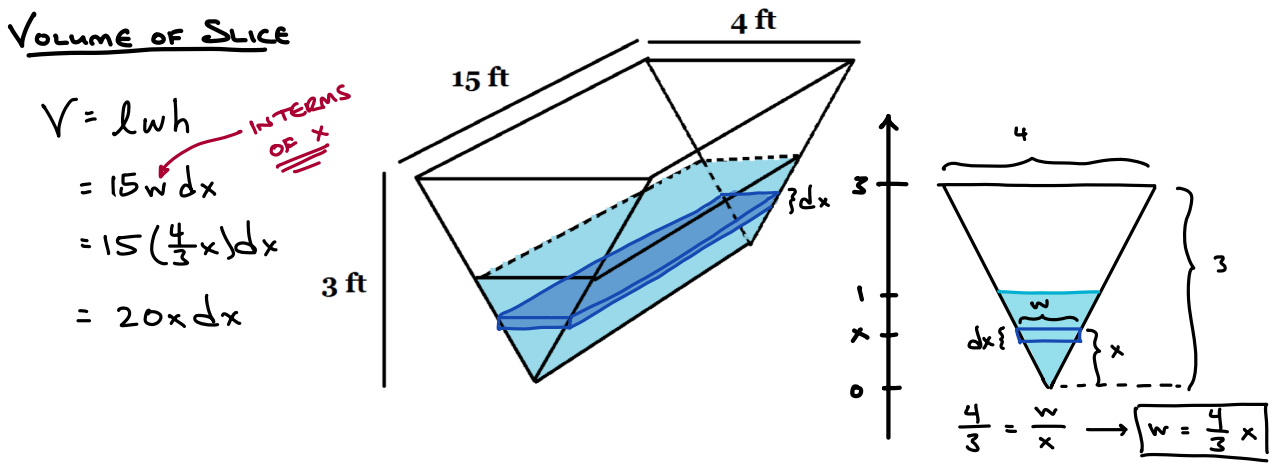


- BUCKET: 5lb
- WATER: $10 - \frac{1}{12}x$ lb
- ROPE: $\frac{1}{2}(30-x)$ lb

$$\int_0^{30} \left[5 + 10 - \frac{1}{12}x + \frac{1}{2}(30-x)\right] dx$$

lb ft

11. The trough pictured below is 15 feet long and 4 feet wide at the top. The ends of the trough are isosceles triangles with a height of 3 feet.



If the trough is filled to a height of 1 foot with water, find the work required to pump all of the water over the side? Recall that water weighs 62.4 lb/ft^3.

Vol → weight

$62.4 \cdot 20x\,dx$

Displacement

$3 - x$

$$\int_0^1 62.4 \cdot 20x \cdot (3-x)\,dx$$

lb/ft · ft² · ft · ft → lb·ft