MATH 122

TRIG SUBSTITUTION

GOALS

- ☐ KNOW WHEN EACH KIND OF TRIG SUB APPLIES
- ☐ USE TRIG INTEGRATION TECHNIQUES TO EVALUATE THESE INTEGRALS

Ex:

$$\int \sqrt{4 - x^2}\, dx$$

LET $x = 2\sin\theta$

LIKE u-SUB, NEED dx:

$$dx = 2\cos\theta\, d\theta$$

$$\int \sqrt{4 - x^2}\, dx = \int \sqrt{4 - (2\sin\theta)^2} \cdot 2\cos\theta\, d\theta$$

$$= \int \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int \sqrt{4(1 - \sin^2\theta)} \cdot 2\cos\theta\, d\theta$$

$$= \int \sqrt{4\cos^2\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int 2\cos\theta \cdot 2\cos\theta\, d\theta$$

$$= 4\int \cos^2\theta\, d\theta$$

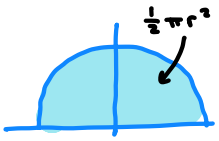
$$= 4 \cdot \frac{1}{2} \int 1 + \cos 2\theta\, d\theta$$

$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

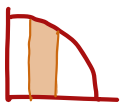
$$= 2\left(\theta + \frac{1}{2} \cdot 2\sin\theta\cos\theta\right) + C$$

EARLIER, WE EVALUATED

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{1}{2}\pi r^2$$



DOESN'T WORK FOR $\int_{1/2}^{1} \sqrt{4 - x^2}\, dx$



$$\begin{bmatrix} \cos^2\theta + \sin^2\theta = 1 \\ 1 + \tan^2\theta = \sec^2\theta \\ 1 + \cot^2\theta = \csc^2\theta \end{bmatrix}$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin A\cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$$

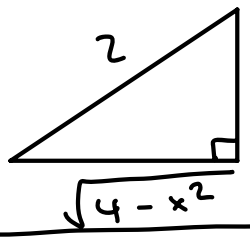
$$\cos A\cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$$

$$\sin A\sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$

$x = 2\sin\theta$

$\frac{x}{2} = \sin\theta$

$\theta = \arcsin\frac{x}{2}$



$\rightarrow \cos\theta = \frac{\sqrt{4-x^2}}{2}$

$$= 2\left(\arcsin\frac{x}{2} + \frac{1}{2} \times \frac{\sqrt{4-x^2}}{2}\right) + C$$

$$= \boxed{2\arcsin\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C}$$

Ex: $\int \frac{1}{x^2\sqrt{16+x^2}}\,dx$ $\qquad x = 4\tan\theta \rightarrow dx = 4\sec^2\theta\,d\theta$

SIMPLIFY $\sqrt{\ }$ - TERM: $\sqrt{16+x^2} = \sqrt{16+(4\tan\theta)^2}$

$= \sqrt{16+16\tan^2\theta}$

$= \sqrt{16(1+\tan^2\theta)}$

$= \sqrt{16\sec^2\theta}$

$= \boxed{4\sec\theta}$

$$\int \frac{1}{x^2\sqrt{16+x^2}}\,dx = \int \frac{1}{(4\tan\theta)^2 \cdot 4\sec\theta} \cdot 4\sec^2\theta\,d\theta$$

$$= \int \frac{1}{16\tan^2\theta}\sec\theta\,d\theta$$

$$= \frac{1}{16}\int \cot^2\theta\sec\theta\,d\theta$$

$$= \frac{1}{16}\int \cot\theta\csc\theta\,d\theta$$

$$= -\frac{1}{16}\csc\theta + C$$

$$= \boxed{-\frac{1}{16}\frac{\sqrt{16+x^2}}{x} + C}$$

$\cot^2\theta\sec\theta$

$\frac{\cos^2\theta}{\sin^2\theta} \cdot \frac{1}{\cos\theta}$

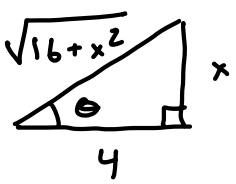
$\frac{\cos\theta}{\sin^2\theta}$

$\frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}$

$\cot\theta\csc\theta$

$\boxed{x = 4\tan\theta}$

$\tan\theta = \frac{x}{4}$



SUMMARY

$\sqrt{}$ - TERM	TRIG - SUB	TRIG - ID	RESULT
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$	$\sqrt{a^2 - x^2} = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$	$\sqrt{a^2 + x^2} = a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$	$\sqrt{x^2 - a^2} = a\tan\theta$

Ex: $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

LET $x = \sec\theta \longrightarrow dx = \sec\theta\tan\theta\,d\theta$

BOUNDS:

$x = 1 \longrightarrow \sec\theta = 1 \longrightarrow \cos\theta = 1 \longrightarrow \theta = 0$

$x = 2 \longrightarrow \sec\theta = 2 \longrightarrow \cos\theta = \frac{1}{2} \longrightarrow \theta = \frac{\pi}{3}$

$$\int_0^{\pi/3} \frac{\tan\theta}{\sec\theta}\cdot \sec\theta\tan\theta\,d\theta$$

$$\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$$

$$\int_0^{\pi/3} \tan^2\theta\,d\theta = \int_0^{\pi/3} \sec^2\theta - 1\,d\theta$$

$$= \tan\theta - \theta \Big|_0^{\pi/3}$$

$$= \left(\sqrt{3} - \frac{\pi}{3}\right) - (0 - 0)$$

$$= \boxed{\sqrt{3} - \frac{\pi}{3}}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin A\cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$$

$$\cos A\cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$$

$$\sin A\sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$

Ex: $\int_{\sqrt{2}}^{2} \frac{\sqrt{4-x^2}}{x^2}\,dx$

$x = 2\sin\theta$ $\quad dx = 2\cos\theta\,d\theta$

Bounds:

$x = \sqrt{2} \longrightarrow \sqrt{2} = 2\sin\theta \longrightarrow \sin\theta = \frac{\sqrt{2}}{2} \longrightarrow \theta = \frac{\pi}{4}$

$x = 2 \longrightarrow 2 = 2\sin\theta \longrightarrow \sin\theta = 1 \longrightarrow \theta = \frac{\pi}{2}$

$$\int_{\pi/4}^{\pi/2} \frac{2\cos\theta}{4\sin^2\theta} \cdot 2\cos\theta\,d\theta$$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta}$$
$$= \sqrt{4(1-\sin^2\theta)}$$
$$= \sqrt{4\cos^2\theta}$$
$$= 2\cos\theta$$

$$\int_{\pi/4}^{\pi/2} \cot^2\theta\,d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta - 1\,d\theta$$

$$= -\cot\theta - \theta \Big|_{\pi/4}^{\pi/2}$$

$$= \left(0 - \frac{\pi}{2}\right) - \left(-1 - \frac{\pi}{4}\right)$$

$$= \boxed{1 - \frac{\pi}{4}}$$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

$\boxed{\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)}$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin A\cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$

$\cos A\cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

Ex: $\int e^x\sqrt{36 - e^{2x}}\,dx$ $\qquad u = e^x \longrightarrow du = e^x\,dx$

$$= \int \sqrt{36 - u^2}\,du.$$

$u = 6\sin\theta \longrightarrow du = 6\cos\theta\,d\theta$

$$\sqrt{36 - 36\sin^2\theta} = \sqrt{36(1-\sin^2\theta)}$$
$$= \sqrt{36\cos^2\theta}$$
$$= 6\cos\theta$$

$$= \int 6\cos\theta \cdot 6\cos\theta\,d\theta$$

$$= 36\int \cos^2\theta\,d\theta = 36\cdot\frac{1}{2}\int 1 + \cos 2\theta\,d\theta$$

$2\sin\theta\cos\theta$

$$= 18\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= 18\left(\theta + \sin\theta\cos\theta\right) + C$$

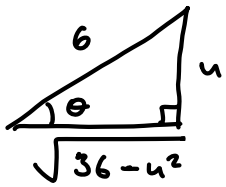
$$= 18\left(\arcsin\frac{u}{6} + \frac{u}{6}\cdot\frac{\sqrt{36-u^2}}{6}\right) + C$$

$$= \boxed{18\arcsin\left(\frac{e^x}{6}\right) + \frac{1}{2}e^x\sqrt{36-e^{2x}} + C}$$

$u = 6\sin\theta$

$\boxed{\sin\theta = \frac{u}{6}}$

$\boxed{\theta = \arcsin\frac{u}{6}}$



$\cos\theta = \frac{\sqrt{36-u^2}}{6}$