MATH 122

SUBSTITUTION

GOALS

- ☐ KNOW HOW TO USE SUBSTITUTION FOR BOTH INDEFINITE AND DEFINITE INTEGRALS
- ☐ RECOGNIZE HELPFUL VS. UNHELPFUL SUBSTITUTION CHOICES

SUBSTITUTION IS LIKE A REVERSE CHAIN RULE

CHAIN RULE: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$

e.g. $y = \sin(x^4) \longrightarrow y' = \underbrace{\cos(x^4)}_{f'(g(x))} \cdot \underbrace{4x^3}_{g'(x)}$

$f(x) = \sin x$; $g(x) = x^4$

How would we compute $\int 4x^3 \cos(x^4)\,dx$?

LET $\boxed{u = x^4}$ ← GOOD STARTING POINT IS TO CHOOSE u = "INSIDE".

GOAL: REWRITE $4x^3\cos(x^4)\,dx$ IN TERMS OF u AND du

* $du \neq dx$ B/C $u = x$ (RATES OF CHANGE ARE DIFFERENT)

$u = x^4 \xrightarrow[\text{DERIVATIVE}]{\text{COMPUTE}} \frac{du}{dx} = 4x^3$ → MULTIPLY BY dx

REWRITE AS $du = 4x^3\,dx$ ← PERFECT MATCH!

$$\int 4x^3 \cos(x^4)\,dx = \int \cos u\,du$$

$$= \sin u + C$$

$$= \boxed{\sin(x^4) + C}$$

Ex: $\int 6x\left(2x^2+1\right)^4\,dx$

LET $u = 2x^2+1$.

TAKE DERN:

$$\frac{du}{dx} = 4x$$

$$du = 4x\,dx$$ ← NOT A PERFECT MATCH!!

OPTION 1: MATCH ONLY X-TERMS

$du = 4x\,dx \longrightarrow \frac{1}{4}du = x\,dx$

PULL OUT CONST.

$$\int 6x\left(2x^2+1\right)^4 dx = \int 6u^4 \frac{1}{4}\,du$$

$$= \frac{3}{2}\int u^4\,du.$$

OPTION 2: SOLVE FOR dx AND SUB.

$du = 4x\,dx \longrightarrow \frac{1}{4x}du = dx$

$$\int 6x\left(2x^2+1\right)^4 dx = \int 6x\,u^4 \cdot \frac{1}{4x}$$

$$= \frac{3}{2}\int u^4\,du.$$

$$= \frac{3}{2}\cdot\frac{1}{5}u^5 + C$$

$$= \boxed{\frac{3}{10}\left(2x^2+1\right)^5 + C}$$

Ex: $\int \frac{4x}{1+x^2}\,dx$

LET $u = 1+x^2$.

THEN $du = 2x\,dx$

$2\,du = 4x\,dx$

$$= 2\int \frac{1}{u}\,du.$$

$$= 2\ln|u| + C$$

$$= \boxed{2\ln|1+x^2| + C}$$

WHEN CHOOSING u:

- MAKE u AN "INSIDE" FUNCTION
  - IN SINE
  - IN ( )
  - IN DENOM.
  - IN EXPONENT
- WANT du TO APPEAR IN THE INTEGRAL GIVE OR TAKE A CONSTANT.

Ex: $\int \frac{4x}{1+x^4}\,dx$

WHY $u = 1 + x^4$ DOESN'T WORK:

$du = 4x^3\,dx$

WE DON'T HAVE AN x^3!

$$\int \frac{4x}{1+x^4}\,dx = \int \frac{4x}{1+(x^2)^2}\,dx$$

$u = x^2 \longrightarrow du = 2x\,dx$

$2\,du = 4x\,dx$

$$= 2\int \frac{1}{1+u^2}\,du$$

$$= 2\arctan u + C$$

$$= \boxed{2\arctan x^2 + C}$$

TRY: $\int \frac{x^2}{\sqrt{1-x^6}}\,dx = \int \frac{x^2}{\sqrt{1-(x^3)^2}}\,dx$

$u = x^3$ $\quad du = 3x^2\,dx$

$\frac{1}{3}\,du = x^2\,dx$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\arcsin u + C$$

$$= \frac{1}{3}\arcsin(x^3) + C$$

CHECK:

$$\left(\tfrac{1}{3}\arcsin x^3\right) = \tfrac{1}{3}\cdot 3x^2 \cdot \frac{1}{\sqrt{1-(x^3)^2}}$$

$$= \frac{x^2}{\sqrt{1-x^6}} \quad \checkmark$$

EX: $\int 4x\sqrt{x+2}\,dx$

NOW WE CAN DISTRIBUTE!

$$= \int 4(u-2)\sqrt{u}\,du$$

$\sqrt{u} = u^{1/2}$

$u = x + 2$

$du = dx$

SOLVE FOR x:

$x = u - 2$

$$= \int 4u^{3/2} - 8u^{1/2}\,du$$

$$= 4\cdot\frac{2}{5}u^{5/2} - 8\cdot\frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{8}{5}(x+2)^{5/2} - \frac{16}{3}(x+2)^{3/2} + C}$$

SADLY, WE CAN'T CHECK EASILY :(

DEFINITE INTEGRALS

Ex: $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$

$u = x^2$; $du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

OPTION 1: KEEP BOUNDS AND GO BACK TO X TO EVALUATE AT END.

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = \frac{1}{2}\int_{x=0}^{x=\sqrt{\pi}} \sin u\,du$$

$$= -\frac{1}{2}\cos u \Big|_{x=0}^{x=\sqrt{\pi}}$$

$$= -\frac{1}{2}\cos x^2 \Big|_0^{\sqrt{\pi}}$$

$$= -\frac{1}{2}\left(\cos\sqrt{\pi}^2 - \cos 0^2\right)$$

$$= -\frac{1}{2}(\cos\pi - \cos 0)$$

$$= -\frac{1}{2}(-1-1)$$

$$= \boxed{1}$$

OPTION 2: CHANGE BOUNDS AND DON'T GO BACK TO X.

$x = 0 \longrightarrow u = 0^2 = 0$

$x = \sqrt{\pi} \longrightarrow u = \sqrt{\pi}^2 = \pi$

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = \frac{1}{2}\int_0^{\pi} \sin u\,du.$$

$$= -\frac{1}{2}\cos u \Big|_0^{\pi}$$

$$= -\frac{1}{2}(\cos\pi - \cos 0)$$

$$= -\frac{1}{2}(-1-1)$$

$$= \boxed{1}$$

EX $\int_1^{e^3} \frac{(\ln x)^2}{x}\,dx$

LET $u = \ln x$ ← GOOD CHOICE B/C IT'S INSIDE A $(\ )^2$ AND ITS DERIV. $\left(\frac{1}{x}\right)$ APPEARS IN THE INTEGRAL

$du = \frac{1}{x}\,dx$

$x = 1 \longrightarrow u = \ln 1 = 0$

$x = e^3 \longrightarrow u = \ln e^3 = 3\ln e = 3$

$$\int_1^{e^3} \frac{(\ln x)^2}{x}\,dx = \int_1^{e^3} (\ln x)^2 \frac{1}{x}\,dx = \int_0^3 u^2\,du.$$

$$= \frac{1}{3}u^3 \Big|_0^3$$

$$= \boxed{9}$$

Ex: $\int \frac{x}{\sqrt{3-x^4}}\,dx = \int \frac{x}{\sqrt{3\left(1-\frac{x^4}{3}\right)}}\,dx$

$= \int \frac{x}{\sqrt{3}\sqrt{1-\frac{x^4}{3}}}\,dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du.$

$= \frac{1}{2}\arcsin u + C$

$= \boxed{\frac{1}{2}\arcsin\left(x^2/\sqrt{3}\right) + C}$

$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}$

goal: $u^2 = \frac{x^4}{3} \rightarrow u = \frac{x^2}{\sqrt{3}}$

$du = \frac{2}{\sqrt{3}}x\,dx$

$\frac{1}{2}du = \frac{1}{\sqrt{3}}x\,dx$