MATH 121

RATES OF CHANGE

GOALS

- ☐ DEFINE AVERAGE RATE OF CHANGE
- ☐ GIVEN A FUNCTION AND TWO x-VALUES, FIND EQUATION OF SECANT LINE
- ☐ USE LIMITS TO DEFINE AND COMPUTE INSTANTANEOUS RATE OF CHANGE
- ☐ GIVE A FUNCTION f AND $x=a$, FIND EQUATION OF TANGENT LINE TO f AT $x=a$
- ☐ GIVEN A PIECEWISE FUNCTIONS WITH SOME UNSPECIFIED PARAMETERS, CHOOSE VALUES FOR THOSE PARAMETERS THAT MAKE THE FUNCTION CONTINUOUS

AVERAGE RATE OF CHANGE.

e.g. IF I DROVE 200 MILES IN 4 HOURS MY AVG. SPEED WAS $\frac{200}{4} = 50$ MILES/HOUR.

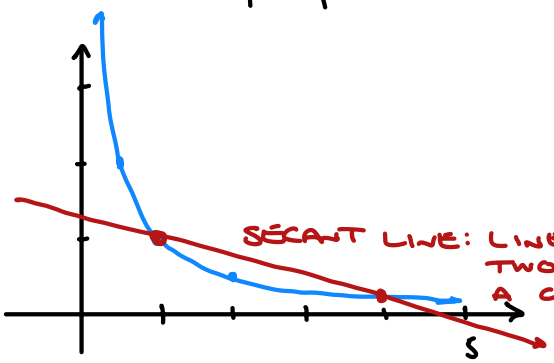
TELLS YOU LITTLE ABOUT MY DRIVING HABITS.

FOR A FUNCTION THE AVG. RATE OF CHANGE FOR $f(x)$ OVER $[a,b]$ IS

$$\frac{f(b)-f(a)}{b-a} = \frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{RISE}}{\text{RUN}}$$

e.g. $f(x) = \frac{1}{x}$; $[1,4]$ ($a=1$, $b=4$)

$$\frac{f(4)-f(1)}{4-1} = \frac{\frac{1}{4}-1}{3} = \frac{-\frac{3}{4}}{3} = \boxed{-\frac{1}{4}}$$



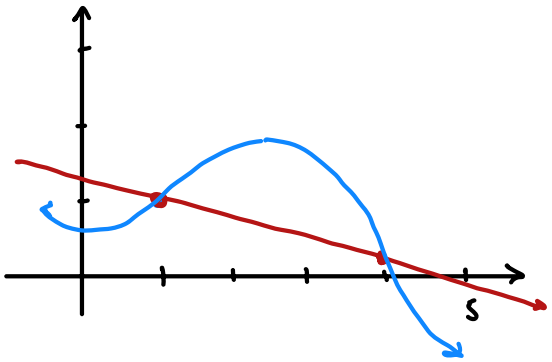
SECANT LINE: LINE THROUGH TWO POINTS ON A GRAPH.

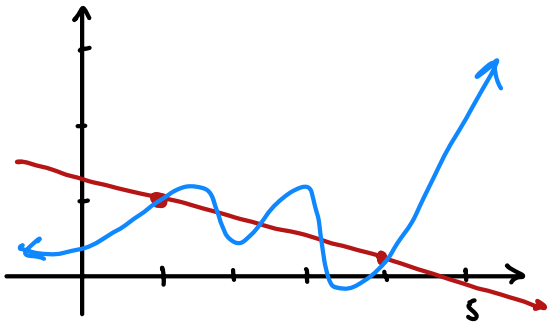
SLOPE OF LINE THROUGH ...

$(1, f(1)) = (1, 1)$

$(4, f(4)) = (4, 1/4)$

* LOTS OF GRAPHS HAVE THIS SAME SECANT LINE





EXAMPLES

1. COMPUTE THE AVG. RATE OF CHANGE, AND WRITE AN EQUATION FOR THE SECANT LINE FOR $f(x) = 2\sqrt{x}$ FROM $x = 1$ TO $x = 4$.

$$m = \frac{f(4) - f(1)}{4 - 1}$$

$$= \frac{2\sqrt{4} - 2\sqrt{1}}{3}$$

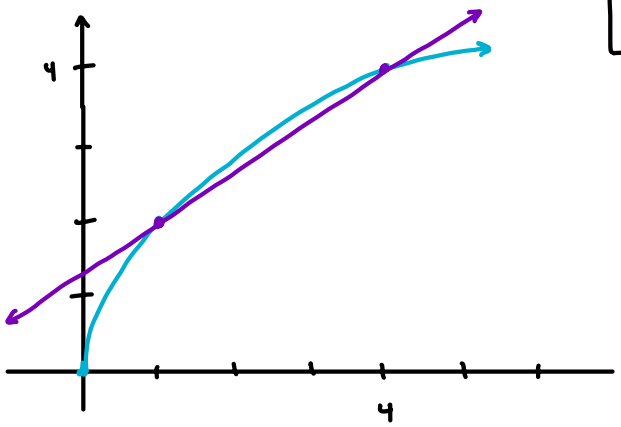
$$= \boxed{\frac{2}{3}}$$

SECANT: $y - y_1 = \frac{1}{2}(x - x_1)$

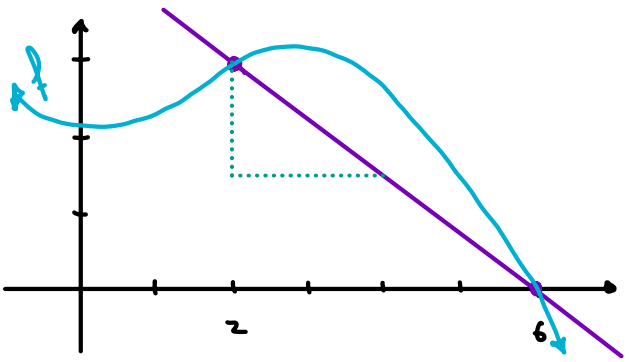
$$y - 2 = \frac{2}{3}(x - 1)$$

$$y = \frac{2}{3}x - \frac{2}{3} + 2$$

$$\boxed{y = \frac{2}{3}x + \frac{4}{3}}$$



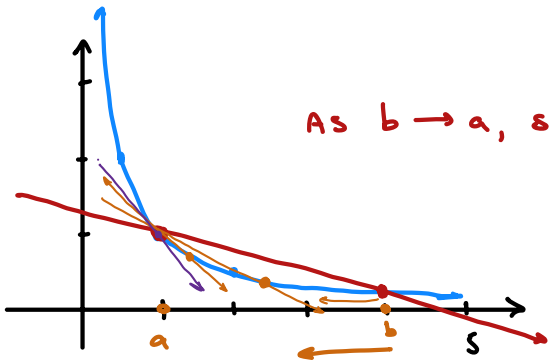
2. DETERMINE THE AVG. RATE OF CHANGE FOR f OVER $[2, 6]$ USING THE GRAPH.



$$\frac{0 - 3}{6 - 2} = \boxed{\frac{-3}{4}}$$

MOVING FROM AVERAGE TO INSTANTANEOUS RATE OF CHANGE:

LET $b \to a$ SO THAT THE LENGTH OF $[a,b]$ GOES TO ZERO (LIMITS!)



AS $b \to a$, SECANTS $\to$ TANGENT.

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

INSTANTANEOUS RATE OF CHANGE FOR f AT $x = a$ (SLOPE OF TANGENT).

e.g $f(x) = \frac{1}{x}$; $\overbrace{x = 1}^{a}$

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a} = \lim_{b \to 1} \frac{f(b) - f(1)}{b - 1}$$

$\frac{0}{0}$ LIMIT (CASE 2)

$$= \lim_{b \to 1} \frac{\frac{1}{b} - 1 \cdot \frac{b}{b}}{b - 1}$$

COMBINE FRACTIONS

$$= \lim_{b \to 1} \frac{\frac{1}{b} - \frac{b}{b}}{b - 1}$$

$$= \lim_{b \to 1} \frac{\frac{1 - b}{b}}{b - 1}$$

$$= \lim_{b \to 1} \frac{1 - b}{b(b - 1)}$$

$$= \lim_{b \to 1} \frac{-(b - 1)}{b(b - 1)}$$

$$= \lim_{b \to 1} -\frac{1}{b}$$

CASE 1

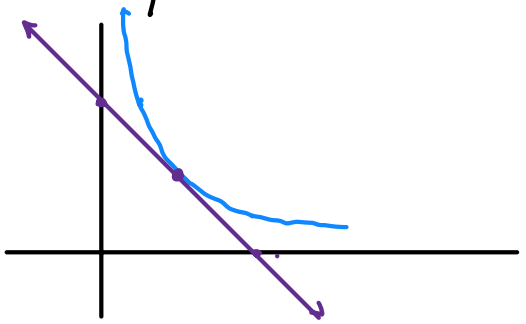
$$= \boxed{-1}$$

EQ: $x = 1$; $y = 1$; $m = -1$

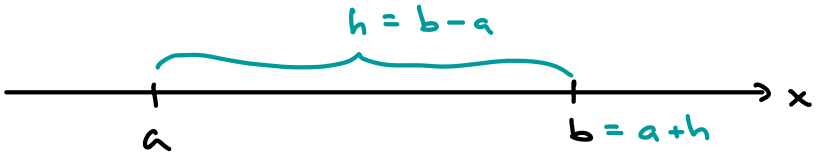
$$y - y_1 = m(x - x_1)$$

$$y - 1 = -1(x - 1)$$

$$y = -x + 2$$



ALTERNATIVE FORMULA: LET $h = b - a$ (or $b = a + h$)



$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

EX: $f(x) = 2x^2 - 4$; $\overbrace{x = 1}^{a}$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f(1+h) = 2(1+h)^2 - 4$$
$$= 2(1 + 2h + h^2) - 4$$
$$= 2 + 4h + 2h^2 - 4$$
$$= 4h + 2h^2 - 2$$

$$f(1) = 2(1)^2 - 4$$
$$= -2$$

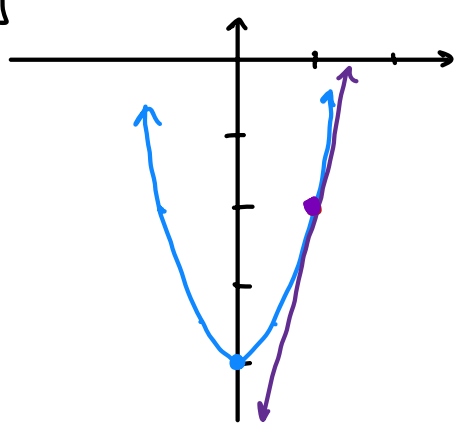
$$m = \lim_{h \to 0} \frac{4h + 2h^2 - 2 - (-2)}{h} = \lim_{h \to 0} \frac{4h + 2h^2}{h} \cdot \frac{1/h}{1/h}$$

$$= \lim_{h \to 0} (4 + 2h)$$

$$= \boxed{4}$$

EQ'N: $x = 1$, $y = f(1) = -2$, $m = 4$

$$y - (-2) = 4(x - 1)$$
$$y = 4x - 4 - 2$$
$$\boxed{y = 4x - 6}$$



Ex: $f(x) = \sqrt{x}$; $x = 4$

$f(4+h) = \sqrt{4+h}$

$f(4) = \sqrt{4} = 2$

$$\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$$

$$= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$\boxed{\frac{1}{4}}$$

CAN'T DISTRIBUTE $\sqrt{\ }$:

e.g. $\sqrt{4+9} \neq 2+3$

$\sqrt{13} \neq 5$

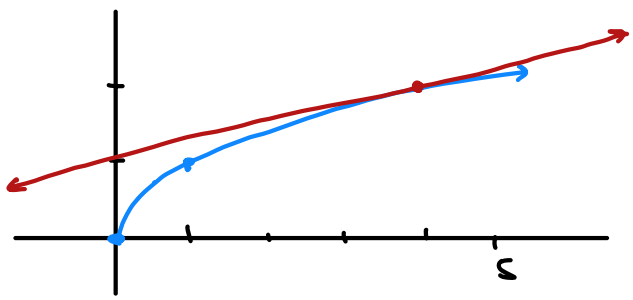
$\frac{a}{a+b} \neq \frac{1}{1+b}$

∴

EQ: $x = 4$, $y = 2$, $m = \frac{1}{4}$

$y - 2 = \frac{1}{4}(x-4)$

$\boxed{y = \frac{1}{4}x + 1}$



Ex: $f(x) = x^2 - 2x$ AT $x = 3$

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$f(x) = (x)^2 - 2(x)$

$f(3+h) = (3+h)^2 - 2(3+h)$

$= 9 + 6h + h^2 - 6 - 2h$

$= \boxed{h^2 + 4h + 3}$

$f(3) = (3)^2 - 2(3) = \boxed{3}$

$$= \lim_{h\to 0} \frac{h^2+4h+3-3}{h}$$

$$= \lim_{h\to 0} \frac{h^2+4h}{h} \quad \frac{1/h}{1/h}$$

$$= \lim_{h\to 0} (h+4)$$

$$= \boxed{4}$$

TANGENT LINE: $x = 3$, $y = f(3) = 3$; $m = 4$

PT - SLOPE

$y - y_1 = m(x - x_1)$

$y - 3 = 4(x - 3)$

$y = 4x - 12 + 3$

$\boxed{y = 4x - 9}$

SLOPE - INT

$y = mx + b$

$3 = 4(3) + b$

$b = -9$

$\boxed{y = 4x - 9}$