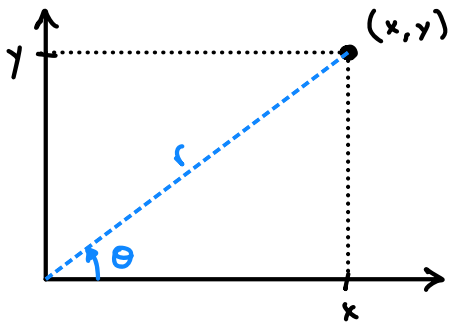
MATH 122

POLAR COORDINATES

GOALS

- [ ] LOCATE POINTS IN POLAR COORDINATES
- [ ] CONVERT BETWEEN POLAR AND RECTANGULAR COORDINATES
- [ ] IDENTIFY BASIC POLAR CURVES: CIRCLES AND CARDIOIDS
- [ ] IDENTIFY POLAR REGIONS AND COMPUTE AREA
- [ ] COMPUTE SLOPES OF TANGENT LINES TO POLAR CURVES
- [ ] COMPUTE ARC LENGTH OF POLAR CURVES

POLAR COORDINATES

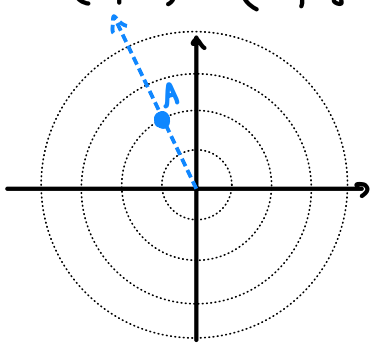


(x, y): RECTANGULAR

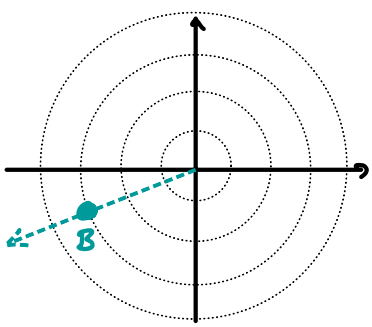
(r, θ): POLAR COORDINATES.

PLOTTING POINTS

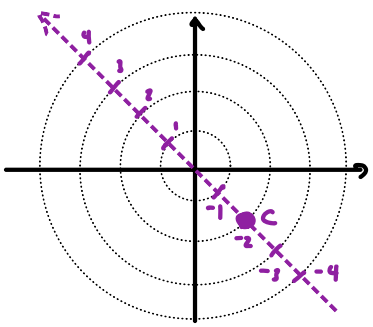
$A(r, \theta) = A\left(2, \frac{2\pi}{3}\right)$



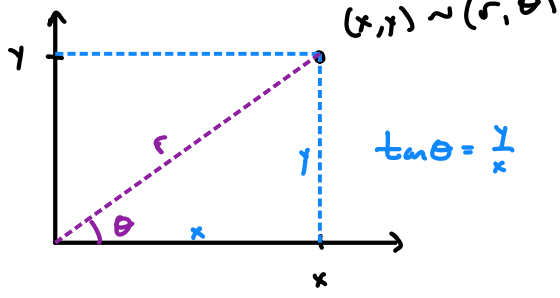
$B(r, \theta) = B\left(3, -\frac{5\pi}{6}\right)$



$C(r, \theta) = C\left(-2, \frac{3\pi}{4}\right)$



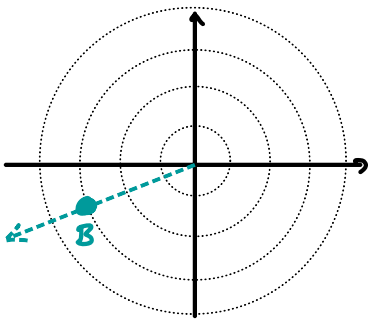
CONVERTING:



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \leftarrow \left(\cos\theta = \frac{x}{r}\right) \quad \leftarrow \left(\sin\theta = \frac{y}{r}\right)$$

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

e.g. $B(r,\theta) = B\left(3, -\frac{5\pi}{6}\right)$

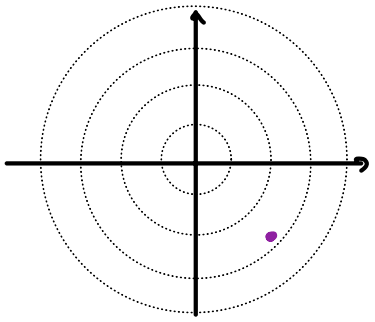


CONVERT TO (x,y):

$$\begin{cases} x = 3\cos(-5\pi/6) = 3\cdot\left(-\frac{\sqrt{3}}{2}\right) \\ y = 3\sin(-5\pi/6) = 3\cdot\left(-\frac{1}{2}\right) \end{cases}$$

$$(x,y) = \left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$$

Ex: CONVERT $(x,y) = (2,-2)$ TO POLAR:



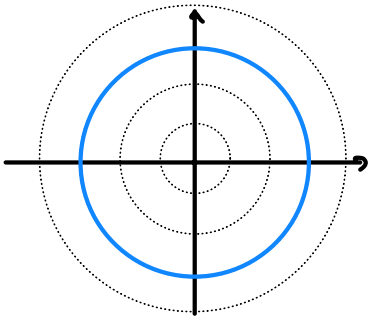
$$\begin{cases} r^2 = 2^2 + (-2)^2 = 8 \longrightarrow r = \pm\sqrt{8} \text{ or } \pm 2\sqrt{2} \\ \tan\theta = \frac{-2}{2} = -1 \longrightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{-\pi}{4}, \frac{-5\pi}{4}, \ldots \end{cases}$$

$$(r,\theta) = \left(-\sqrt{8}, \frac{3\pi}{4}\right), \left(\sqrt{8}, \frac{7\pi}{4}\right), \left(\sqrt{8}, \frac{-\pi}{4}\right), \ldots$$

GRAPHS:

r = CONST

e.g. $r = 3$

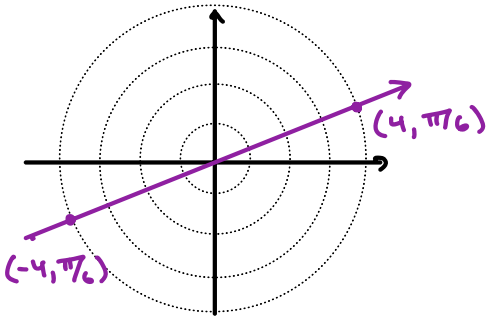


$r = 3 \longrightarrow r^2 = 9$

$\boxed{x^2 + y^2 = 9}$

θ = CONST

e.g. $\theta = \frac{\pi}{6}$



$\theta = \frac{\pi}{6} \longrightarrow \tan\theta = \tan\frac{\pi}{6}$

$\frac{y}{x} = \frac{1}{\sqrt{3}}$

$\boxed{y = \frac{1}{\sqrt{3}}x}$

CIRCLES TANGENT AT ORIGIN

e.g. $r = 2\cos\theta$

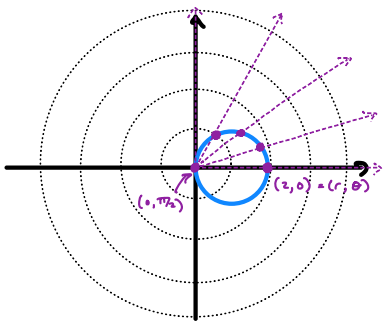
$r^2 = 2r\cos\theta$ MULT. BY r

$x^2 + y^2 = 2x$ COMPLETE THE SQ !!

$x^2 - 2x + y^2 = 0$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$ RADIUS: 1 ; CENTER: (1, 0)

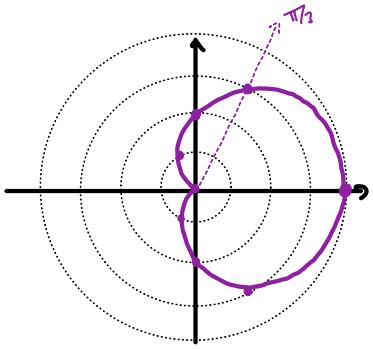


$r = 2\cos\theta$

θ	r
0	2
$\pi/3$	1
$\pi/2$	0

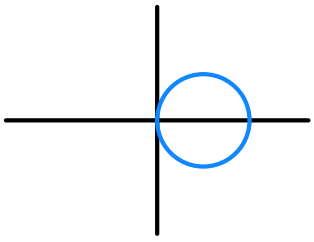
CARDIOID:

e.g. $r = 2 + 2\cos\theta$ $\longrightarrow$ $x^2 + y^2 = 2\sqrt{x^2+y^2} + 2x$

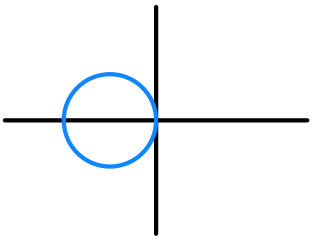


θ	r
0	4
$\pi/3$	3
$\pi/2$	2
$2\pi/3$	1
π	0

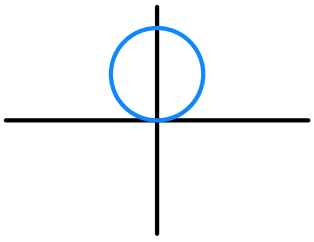
VARIATIONS:



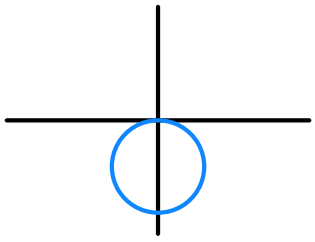
$r = a\cos\theta$



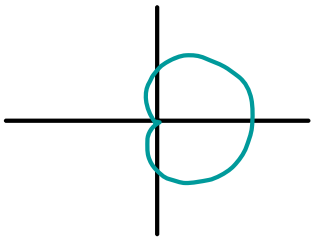
$r = -a\cos\theta$



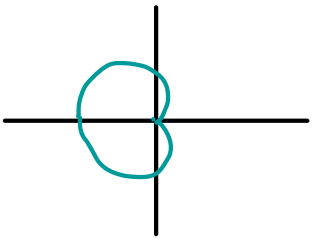
$r = a\sin\theta$



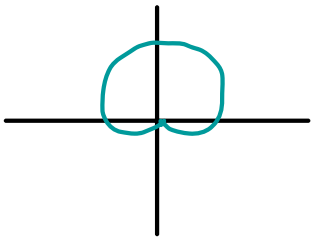
$r = -a\sin\theta$



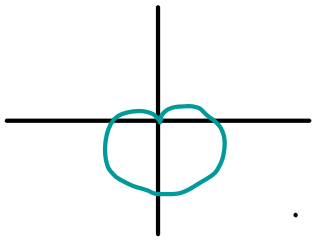
$r = a + a\cos\theta$



$r = a - a\cos\theta$



$r = a + a\sin\theta$



$r = a - a\sin\theta$

TANGENT LINES: SLOPE IS GIVEN BY $\frac{dy}{dx}$

COMBINE TWO IDEAS:

① $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

② COMPUTING $\frac{dy}{dx}$ FROM PARAMETRIC EQNS

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

* DO THIS W/ θ IN PLACE OF t

FORMULA

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

PRODUCT RULE FOR $y = r\sin\theta$ AND $x = r\cos\theta$

Ex: $r = 2\sin\theta$; compute tangent line at $\theta = \frac{\pi}{6}$.

Method 1: use formula

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

$$\boxed{r' = 2\cos\theta}$$

$$= \frac{2\cos\theta\sin\theta + 2\sin\theta\cos\theta}{2\cos\theta\cos\theta - 2\sin\theta\sin\theta}$$

$$= \boxed{\frac{2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta}}$$

Method 2: plug r into conversion

$$\begin{cases} x = r\cos\theta \longrightarrow 2\underbrace{\sin\theta}_{A}\underbrace{\cos\theta}_{B} \\ y = r\sin\theta \longrightarrow 2\sin\theta\sin\theta = 2\sin^2\theta \end{cases}$$

$2(\sin\theta)^2$

$$\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{\underset{A'}{2\cos\theta}\underset{B}{\cos\theta} - \underset{A}{2\sin\theta}\underset{B'}{\sin\theta}}$$

$$= \boxed{\frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}}$$

Tangent line:

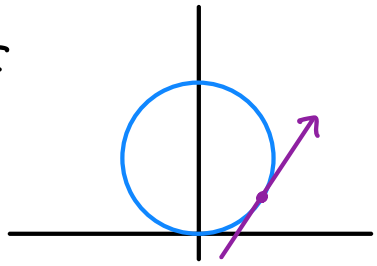
Slope: $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{2\sin\frac{\pi}{6}\cos\frac{\pi}{6}}{\cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6}} = \frac{2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}}{\frac{3}{4} - \frac{1}{4}}$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= \boxed{\sqrt{3}}$$

Point:
$$\begin{cases} x = 2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \\ y = 2\sin^2\frac{\pi}{6} = 2\cdot\frac{1}{4} = \frac{1}{2} \end{cases}$$

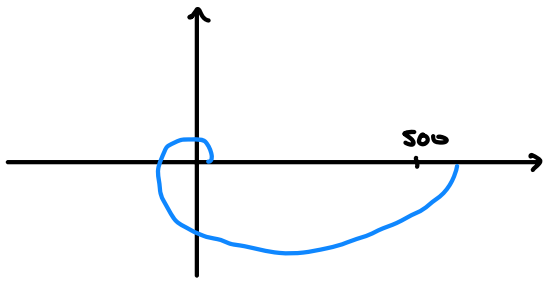
$$\boxed{y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)}$$



ARC LENGTH:

① $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ ② $\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ ③ $\int_\alpha^\beta \sqrt{r^2 + (r')^2}\,d\theta$

e.g. $r = e^\theta \quad [0, 2\pi]$



$r' = e^\theta$

$$L = \int_0^{2\pi} \sqrt{(e^\theta)^2 + (e^\theta)^2}\,d\theta$$

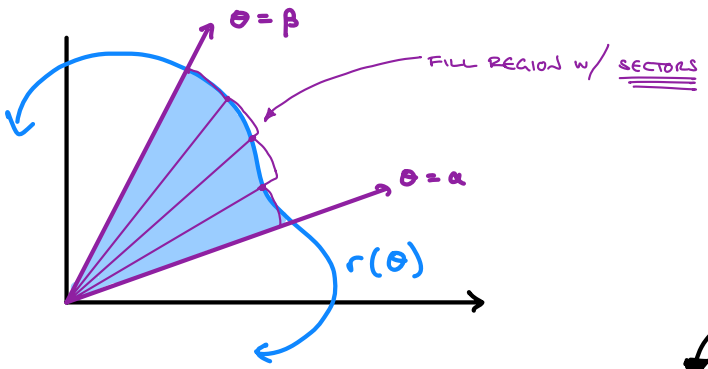
$$= \int_0^{2\pi} \sqrt{2e^{2\theta}}\,d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\,e^\theta\,d\theta$$

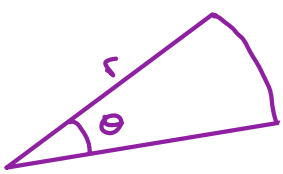
$$= \sqrt{2}\,e^\theta \Big|_0^{2\pi}$$

$$= \sqrt{2}\left(e^{2\pi} - 1\right)$$

AREA OF POLAR REGIONS



AREA OF A SECTOR: $A = \frac{1}{2}\theta r^2$



$$A = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta$$

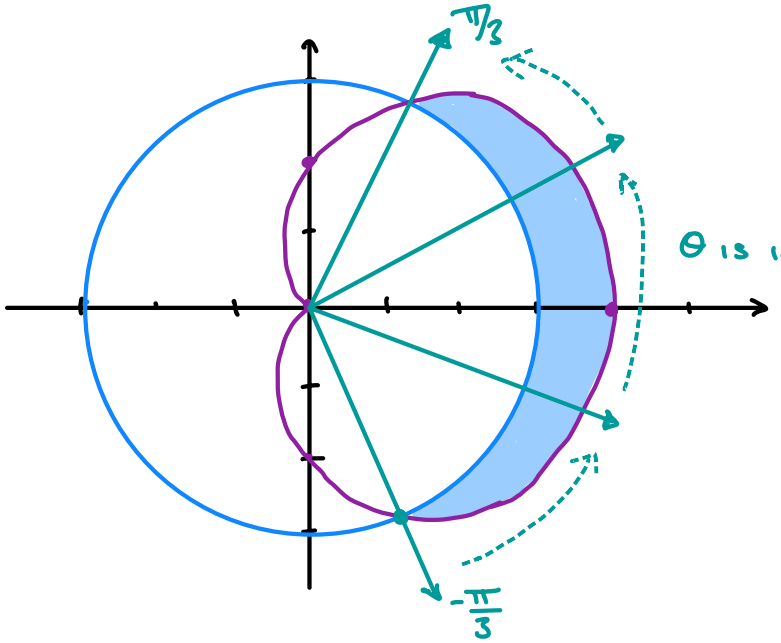
EX: CHECK WITH $r = 2$ AND $[0, 2\pi]$ (SHOULD BE 4π)

$$A = \int_0^{2\pi} \frac{1}{2}(2)^2 \, d\theta$$

$$= 2\theta \Big|_0^{2\pi}$$

$$= \boxed{4\pi} \checkmark$$

EX FIND AREA OUTSIDE $r = 3$ AND INSIDE $r = 2 + 2\cos\theta$



INTERSECTIONS:

$$2 + 2\cos\theta = 3$$

$$2\cos\theta = 1$$

$$\cos\theta = 1/2$$

$$\theta = \frac{\pi}{3}, \underline{-\frac{\pi}{3}}$$

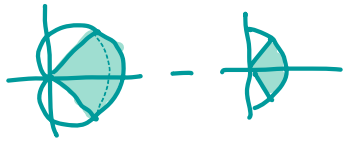
EVEN THOUGH $\frac{5\pi}{3}$ POINTS TO SAME LOCATION, WE NEED BOUNDS IN ORDER.

COULD DO $\int_{5\pi/3}^{7\pi/3}$ —

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2+2\cos\theta)^2 - (3)^2 \, d\theta$$

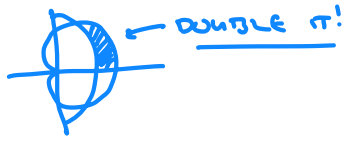
$r_{out}^2 \quad - \quad r_{in}^2$

$\leftarrow \left(\frac{1}{2}r_{out}^2 \, d\theta - \frac{1}{2}r_{in}^2 \, d\theta\right)$



USE SYMMETRY

$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/3} (2+2\cos\theta)^2 - 3^2 \, d\theta$$



$$= \int_0^{\pi/3} 4 + 8\cos\theta + 4\underbrace{\cos^2\theta}_{\text{DOUBLE ANGLE ID}} - 9 \, d\theta$$

$$= \int_0^{\pi/3} 8\cos\theta + 4\cdot\tfrac{1}{2}(1+\cos 2\theta) - 5 \, d\theta$$

$$2 + 2\cos 2\theta$$

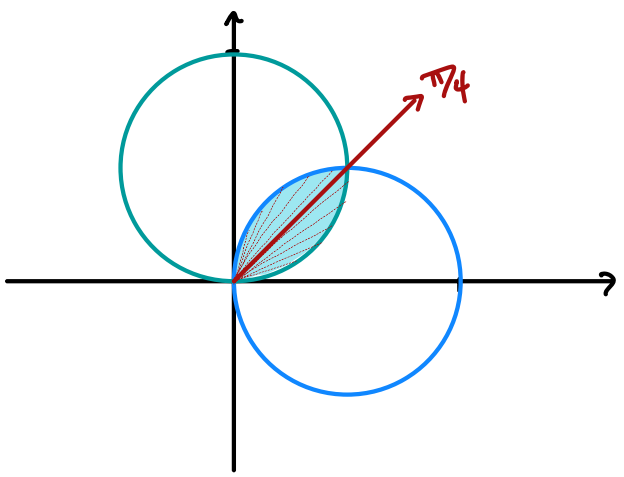
$$= \int_0^{\pi/3} 8\cos\theta + 2\cos 2\theta - 3 \, d\theta$$

$$= 8\sin\theta + \sin 2\theta - 3\theta \Big|_0^{\pi/3}$$

$$= 8\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - 3\cdot\frac{\pi}{3} - (0+0-0)$$

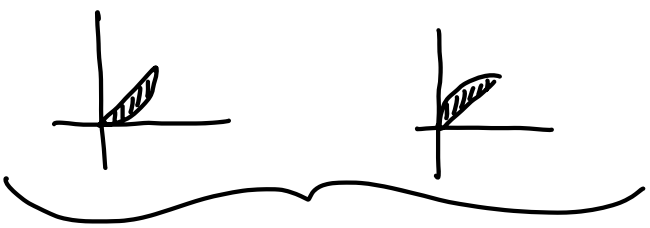
$$= \boxed{\frac{9\sqrt{3}}{2} - \pi}$$

EX: AREA ENCLOSED BY BOTH $\boxed{r = \cos\theta}$ AND $\boxed{r = \sin\theta}$.



INTERSECTION: $\cos\theta = \sin\theta \longrightarrow \theta = \frac{\pi}{4}$

$$A = \frac{1}{2}\int_0^{\pi/4} (\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} (\cos\theta)^2 d\theta$$



USE SYMMETRY

$$= 2\cdot\frac{1}{2}\int_0^{\pi/4} (\sin\theta)^2 d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} - \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \Big|_0^{\pi/4}$$

$$= \boxed{\frac{\pi}{8} - \frac{1}{4}}$$