MATH 121

LIMITS OF TRIG FUNCTIONS

GOALS

- ☐ KNOW BASIC LIMITS FOR ALL SIX TRIG FUNCTIONS
- ☐ APPLY THE SQUEEZE THEOREM TO COMPLEX LIMITS
- ☐ DERIVE AND APPLY $\lim_{x \to 0} \frac{\sin x}{x}$
- ☐ USE SUBSTITUTION TO SIMPLIFY COMPLEX LIMITS

BASIC TRIG LIMITS

$\sin x$ & $\cos x$

SINCE $\sin x$ AND $\cos x$ ARE CONTINUOUS EVERYWHERE...

$$\lim_{x \to a} \sin x = \sin a \qquad \lim_{x \to a} \cos x = \cos a.$$

$$\text{e.g. } \lim_{x \to -\frac{\pi}{3}} \sin x = \sin\left(-\frac{\pi}{3}\right) = \boxed{-\frac{\sqrt{3}}{2}}$$

BUT, SINCE $\sin x$ AND $\cos x$ BOUNCE BETWEEN -1 AND 1...

$$\lim_{x \to \pm\infty} \sin x \text{ DNE} \quad \& \quad \lim_{x \to \pm\infty} \cos x \text{ DNE}$$

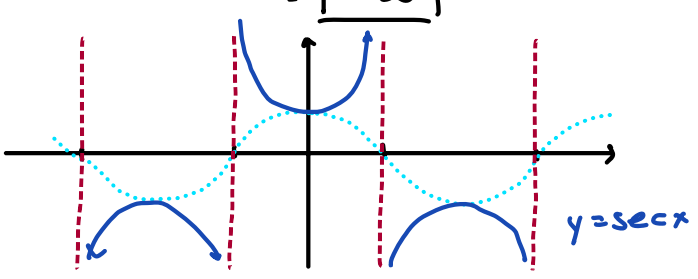
THE OTHER FOUR

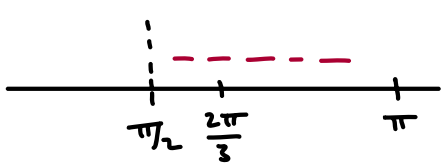
$\tan x$, $\cot x$, $\sec x$, AND $\csc x$ ALL HAVE VERTICAL ASYMPTOTES SO WE CAN USE OUR APPROACH TO "CASE 3" LIMIT $(K/0)$

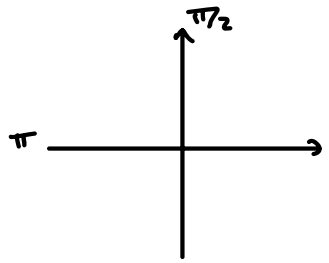
EXAMPLES

1. $\lim_{x \to \pi/2^+} \sec x = \lim_{x \to \pi/2^+} \frac{1 \to 1}{\cos x \to 0}$

$= \boxed{-\infty}$



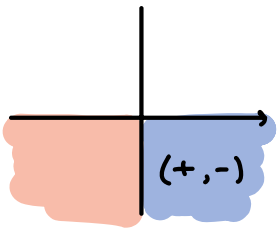




2. $\lim_{x \to 0^-} \cot x = \lim_{x \to 0^-} \frac{\cos x}{\sin x}$ ($\cos x \to 1$, $\sin x \to 0$)

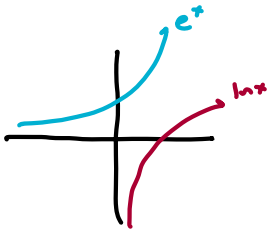
LESS THAN ZERO

$= \boxed{-\infty}$



3. $\lim_{x \to \pi^+} e^{\csc x} = \lim_{t \to -\infty} e^t$

$= \boxed{0}$

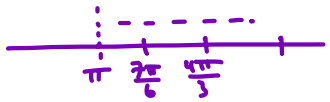


LET $t = \csc x$

$\lim_{x \to \pi^+} \csc x = \lim_{x \to \pi^+} \frac{1}{\sin x}$ ($\sin x \to 0$, $\frac{1}{\sin x} \to 1$)

$= \boxed{-\infty}$

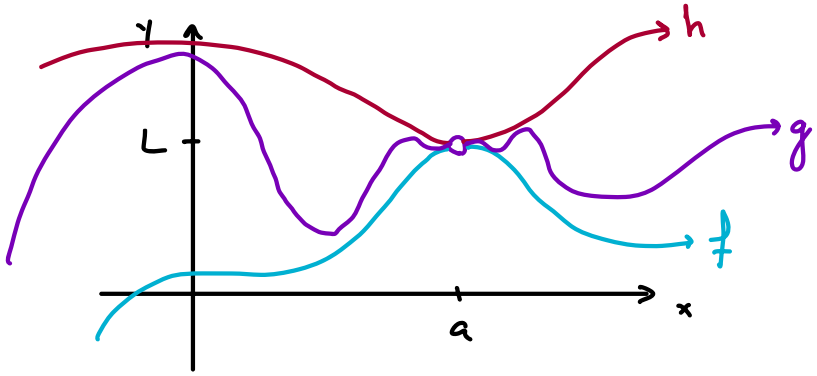
so $t \to -\infty$



THE SQUEEZE THEOREM

IF $f(x) \le g(x) \le h(x)$ AND THEY'RE ALL CONTINUOUS, $(x \neq a)$

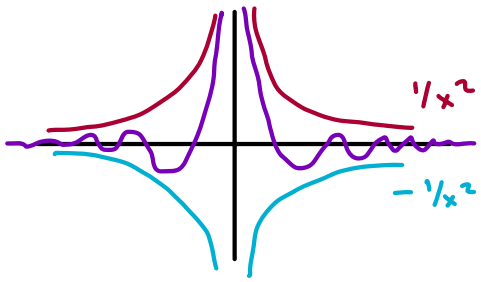
IF $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ THEN $\lim_{x \to a} g(x) = L$



e.g. $\lim_{x \to +\infty} \frac{1}{x^2} \cos(x^2) \longrightarrow 0 \cdot \text{DNE}$

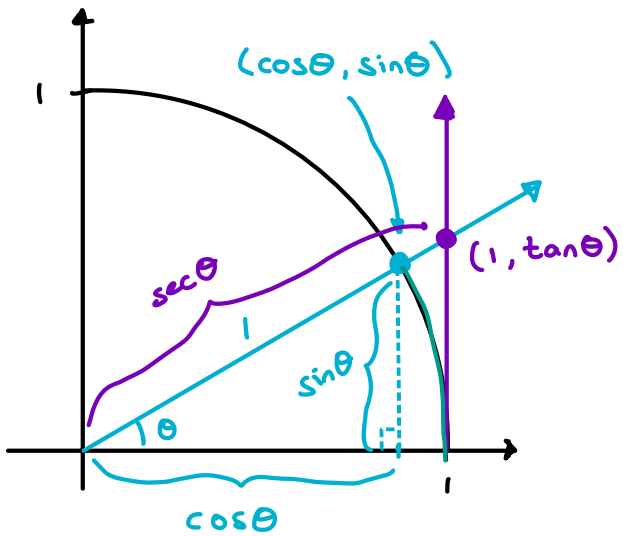
$-1 \le \cos x^2 \le 1$

$-\frac{1}{x^2} \le \frac{1}{x^2} \cos x^2 \le \frac{1}{x}$



$\lim_{x \to +\infty} -\frac{1}{x^2} = 0$ AND $\lim_{x \to +\infty} \frac{1}{x^2} = 0$ SO $\lim_{x \to \infty} \frac{1}{x^2} \cos x^2 = 0$

DERIVING $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ ($\sin\theta \to 0$, $\theta \to 0$)

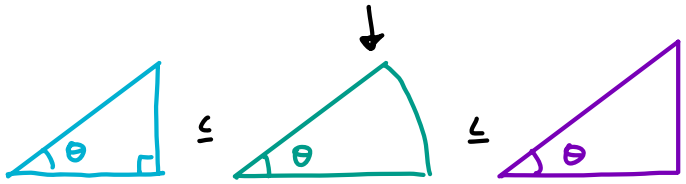


$\cos^2\theta + \sin^2\theta = 1$

$\div \cos\theta$

$1 + \tan^2\theta = \sec^2\theta$

AREA OF SECTOR: $A = \frac{1}{2}\theta r^2$



AREAS

$$\frac{1}{2}\cos\theta\sin\theta \le \frac{1}{2}\theta(1)^2 \le \frac{1}{2}(1)\tan\theta$$

$$\cos\theta\sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta} \quad (\div \sin\theta)$$

$$\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

$$\frac{1}{\cos\theta} \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$

$\lim_{\theta \to 0} \frac{1}{\cos\theta} = 1$ AND $\lim_{\theta \to 0} \cos\theta = 1$ SO...

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

EXAMPLES

1. $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta}$ ($1 - \cos\theta \to 0$, $\theta \to 0$)

MULT. BY CONJUGATE

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta} = \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)}$$

$\cos^2\theta + \sin^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1 + \cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1 + \cos\theta}$$

$$= (1) \cdot \frac{0}{1 + 1}$$

$$= \boxed{0}$$

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

2. $\lim_{x\to 0} \frac{\sin(5x)}{2x} \cdot \frac{2.5}{2.5} = \lim_{x\to 0} 2.5 \cdot \frac{\sin 5x}{5x}$

$\theta = 5x$

NOT θ

$= 2.5$

$= \boxed{\frac{5}{2}}$

OR

$\lim_{x\to 0} \frac{\sin(5x)}{2x} \cdot \frac{5}{5}$

$= \lim_{x\to 0} \frac{5}{2} \cdot \frac{\sin 5x}{5x}$

$= \boxed{\frac{5}{2}}$

3. $\lim_{x\to 0} \frac{\sin^2(5x)}{2x^2} = \lim_{x\to 0} \frac{1}{2} \cdot \frac{\sin(5x)}{x} \cdot \frac{\sin(5x)}{x} \cdot \frac{5}{5} \cdot \frac{5}{5}$

$= \lim_{x\to 0} \left(\frac{25}{2}\right)\left(\frac{\sin(5x)}{5x}\right)\left(\frac{\sin(5x)}{5x}\right)$

$= \frac{25}{2} \cdot 1 \cdot 1$

$= \boxed{\frac{25}{2}}$

4. $\lim_{x\to 0} \frac{\tan 2x}{\sin 3x} = \lim_{x\to 0} \frac{\sin(2x)}{\cos(2x)\sin(3x)} \cdot \frac{2}{2} \cdot \frac{3}{3} \cdot \frac{x}{x}$

$= \lim_{x\to 0} \frac{1}{\cos(2x)} \cdot \frac{\sin(2x)}{2x} \cdot \frac{3x}{\sin(3x)} \cdot \frac{2}{3}$

$= \left(\frac{1}{1}\right)(1)(1)\left(\frac{2}{3}\right)$

$= \boxed{2/3}$

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$

$* \frac{1}{1} = 1$

$\cos(0) = 1$

5. $$\lim_{x \to -\infty} \frac{\sin(e^x)}{e^x} = \lim_{t \to 0} \frac{\sin t}{t} = \boxed{1}$$

SUBSTITUTION:

$t = e^x$

AS $x \to -\infty$,

$e^x \to 0$

$e^{2x} = (e^x)^2$

6. $$\lim_{x \to -\infty} \frac{\sin(e^{2x})}{e^x} = \lim_{x \to -\infty} \frac{\sin[(e^x)^2]}{e^x}$$

$$= \lim_{t \to 0} \frac{\sin(t^2)}{t} \cdot \frac{t}{t}$$

$$= \lim_{t \to 0} \frac{\sin(t^2)}{t^2} \cdot t$$

$$= (1)(0)$$

$$= \boxed{0}$$