MATH 121

LIMITS AT INFINITY

GOALS

- ☐ DEFINE LIMITS AT INFINITY / END BEHAVIOR
- ☐ USE END BEHAVIOR OF $\frac{1}{x}$ TO DERIVE END BEHAVIOR OF POLYNOMIALS AND RATIONAL FUNCTIONS
- ☐ DEFINE HORIZONTAL ASYMPTOTES IN TERMS OF END BEHAVIOR
- ☐ APPLY THE "LEADING TERM TRICK" TO COMPUTE END BEHAVIOR OF POLYNOMIALS, RATIONAL FUNCTIONS, AND OTHER SPECIAL CASES.

END BEHAVIOR OF A FUNCTION

IF WE CAN MAKE $f(x)$ AS CLOSE TO SOME FINITE VALUE L AS WE LIKE BY MAKING x LARGE ENOUGH, FOR $x > 0$, WE WRITE $\lim_{x \to +\infty} f(x) = L$

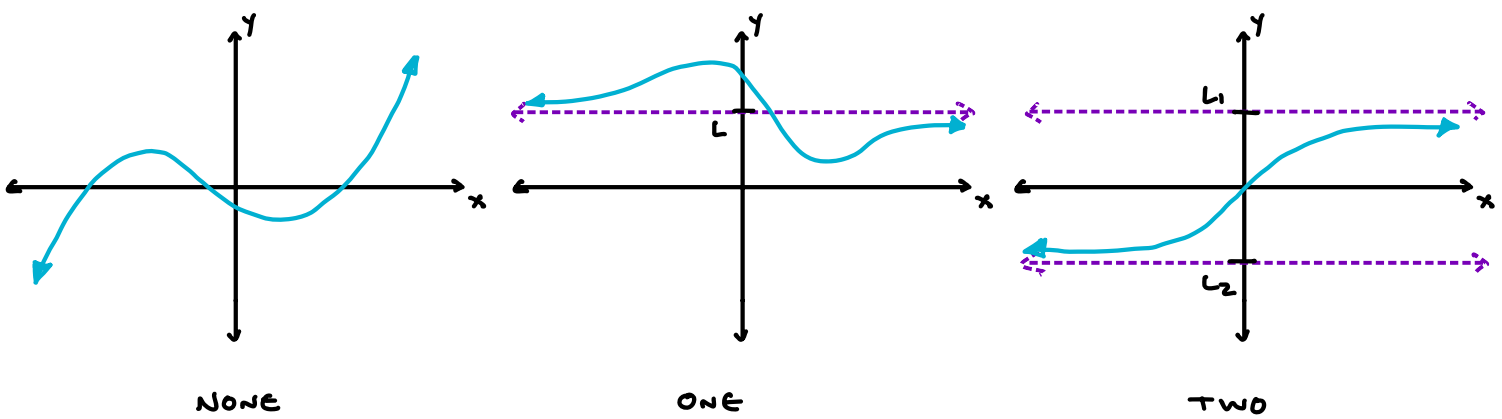
IF WE CAN MAKE $f(x)$ AS CLOSE TO SOME FINITE VALUE L AS WE LIKE BY MAKING x LARGE ENOUGH, FOR $x < 0$, WE WRITE $\lim_{x \to -\infty} f(x) = L$

* THESE ARE LIMITS AT INFINITY. ALSO CALLED THE END BEHAVIOR OF f *

HORIZONTAL ASYMPTOTES

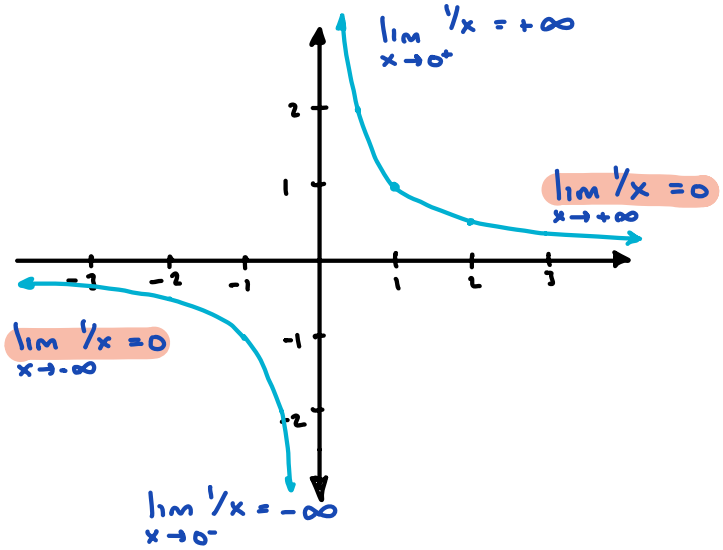
IF $\lim_{x \to +\infty} f(x) = L$ OR $\lim_{x \to -\infty} f(x) = L$, $y = L$ IS A HORIZONTAL ASYMPTOTE.

A FUNCTION CAN HAVE AT MOST TWO HORIZONTAL ASYMPTOTES...



NONE ONE TWO

OUR STARTING POINT: $f(x) = \frac{1}{x}$



x	1	2	10	100	1000	$\longrightarrow +\infty$
$f(x)$	1	$1/2$	$1/10$	$1/100$	$1/1000$	$\longrightarrow 0$

x	-1	-2	-10	-100	-1000	$\longrightarrow -\infty$
$f(x)$	-1	$-1/2$	$-1/10$	$-1/100$	$-1/1000$	$\longrightarrow 0$

ASYMPTOTES:

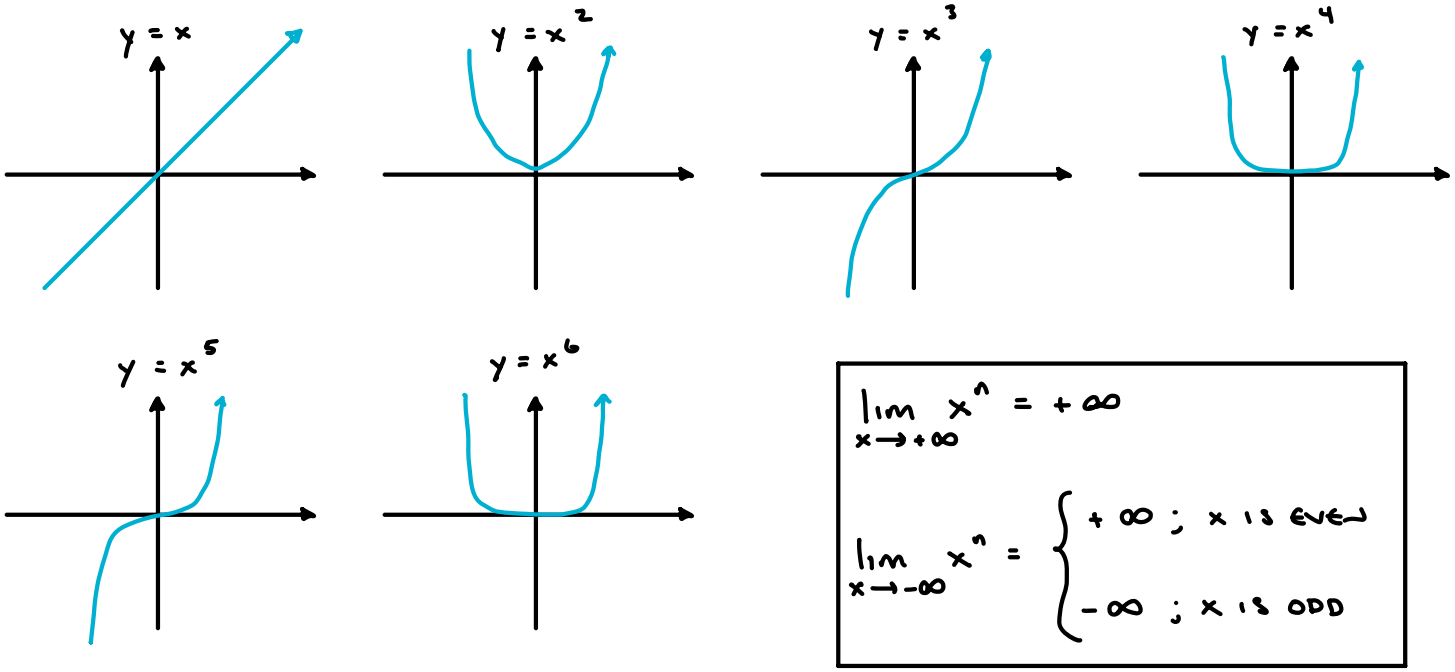
- HORIZONTAL: $y = 0$
- VERTICAL: $x = 0$

EXTENDING THIS LIMIT: SINCE $\lim_{x\to\infty} \frac{1}{x} = 0$, IT FOLLOWS $\lim_{x\to\infty} \frac{1}{x^n} = 0$ FOR $n > 0$

LIMITS OF POWER FUNCTIONS

DEF: A POWER FUNCTION IS A FUNCTION OF THE FORM $f(x) = x^n$, WHERE n IS A POSITIVE INTEGER.

LOOKING FOR A PATTERN



$$\lim_{x\to+\infty} x^n = +\infty$$

$$\lim_{x\to-\infty} x^n = \begin{cases} +\infty \text{ ; } x \text{ IS EVEN} \\ -\infty \text{ ; } x \text{ IS ODD} \end{cases}$$

EXAMPLES

1. $\lim_{x \to +\infty} x^{11} = +\infty$

2. $\lim_{x \to -\infty} x^{11} = -\infty$

3. $\lim_{x \to +\infty} x^{20} = +\infty$

4. $\lim_{x \to -\infty} x^{20} = +\infty$

LIMITS OF POLYNOMIALS

DEF A POLYNOMIAL IS A FUNCTION OF THE FOLLOWING FORM

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \; ; \; n \text{ IS A POSITIVE INTEGER}$$

APPLYING WHAT WE KNOW

$$\lim_{x \to \infty} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

$$= \lim_{x \to \infty} x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right) \quad \text{FACTOR OUT } x^n$$

$$= \lim_{x \to \infty} x^n \left(a_n + 0 + \cdots + 0 + 0 + 0 \right) \quad \text{APPLY } \lim_{x \to \infty} \frac{1}{x} = 0$$

$$= \boxed{\lim_{x \to \infty} a_n x^n}$$

SUMMARY IF $x \to \pm\infty$, THE **LEADING TERM** DETERMINES THE LIMIT OF A POLYNOMIAL

EXAMPLES

1. $\lim_{x \to \infty} 3x^4 - 7x^2 + 5x + 1 = \lim_{x \to \infty} 3x^4$

$$= \boxed{+\infty}$$

2. $\lim_{x \to -\infty} 5 + x - x^3 = \lim_{x \to -\infty} (-x^3)$

$$= -(-\infty)$$

$$= \boxed{+\infty}$$

LIMITS OF RATIONAL FUNCTIONS

DEF A RATIONAL FUNCTION IS ONE POLYNOMIAL DIVIDED BY ANOTHER

$$r(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0}$$

APPLYING WHAT WE KNOW TAKE THE LEADING TERMS

$$\lim_{x \to \infty} r(x) = \lim_{x \to \infty} \frac{a_n x^n}{b_m x^m}$$

EXAMPLES

1. $$\lim_{x \to -\infty} \frac{5x^4 + 7x - 1}{3x^5 + 2x^4 + 3} = \lim_{x \to -\infty} \frac{5x^4}{3x^5}$$ LEADING TERMS

$$= \lim_{x \to -\infty} \frac{5}{3x}$$ SIMPLIFY

$$= \boxed{0}$$ APPLY $\lim_{x \to \infty} \frac{1}{x} = 0$

2. $$\lim_{x \to +\infty} \frac{x^2 + 5x + 6}{4x^2 + x - 1} = \lim_{x \to +\infty} \frac{x^2}{4x^2}$$ LEADING TERMS

$$= \lim_{x \to +\infty} \frac{1}{4}$$ SIMPLIFY

$$= \boxed{\frac{1}{4}}$$ APPLY $\lim_{x \to a} k = k$

3. $$\lim_{x \to -\infty} \frac{x^6 - 1}{x^3 + 2x + 5} = \lim_{x \to -\infty} \frac{x^6}{x^3}$$

$$= \lim_{x \to -\infty} x^3$$

$$= \boxed{-\infty}$$

OTHER CASES

1. $$\lim_{x \to -\infty} \frac{\sqrt{5x^2+1}}{3-x} = \lim_{x \to -\infty} \frac{\sqrt{5x^2}}{-x}$$

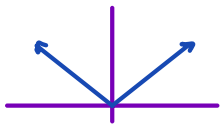
$$= \lim_{x \to -\infty} \frac{\sqrt{5}\,|x|}{-x}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{5}\,(-x)}{-x}$$

$$= \lim_{x \to -\infty} \sqrt{5}$$

$$= \boxed{\sqrt{5}}$$

$$|x| = \begin{cases} x\,; & x \geq 0 \\ -x\,; & x < 0 \end{cases}$$



2. $$\lim_{x \to -\infty} \sqrt{x^2+3x} + x \longrightarrow \infty - \infty$$

INDETERMINATE FORM → GET INTO FRACTION FORM.

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+3x} + x}{1} \cdot \frac{\sqrt{x^2+3x} - x}{\sqrt{x^2+3x} - x}$$

$$= \lim_{x \to -\infty} \frac{x^2+3x-x^2}{\sqrt{x^2+3x} - x}$$

$$= \lim_{x \to -\infty} \frac{3x}{\sqrt{x^2+3x} - x}$$

APPLY LEADING TERM TRICK.

$$= \lim_{x \to -\infty} \frac{3x}{\sqrt{x^2} - x}$$

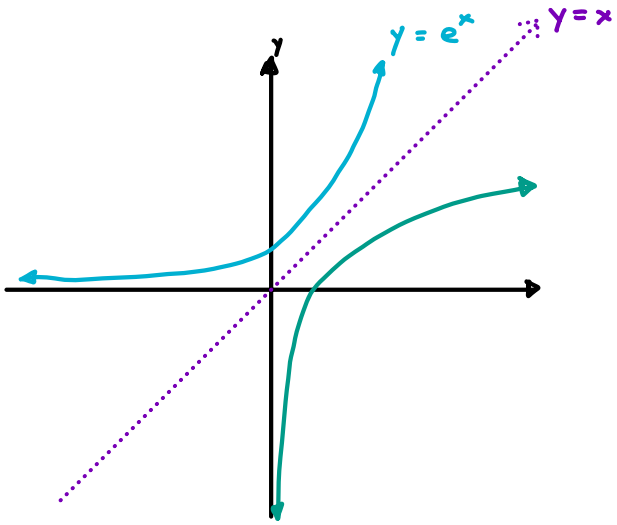
$$= \lim_{x \to -\infty} \frac{3x}{|x| - x}$$

$$= \lim_{x \to -\infty} \frac{3x}{-x-x}$$

$$= \lim_{x \to -\infty} \frac{3x}{-2x}$$

$$= \boxed{-3/2}$$

LOGS + EXPONENTIALS



$$\lim_{x \to +\infty} e^x = +\infty$$

$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to +\infty} \ln x = +\infty$$

$$\lim_{x \to 0^+} \ln x = -\infty$$

EXAMPLES

1. $\lim_{x \to \infty} \dfrac{3}{e^x + 1}$ $\quad 3 \longrightarrow$, $e^x + 1 \longrightarrow \infty$

$$= \boxed{0}$$

2. $\lim_{x \to \infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $\quad e^x - e^{-x} \longrightarrow \infty$, $e^x + e^{-x} \longrightarrow \infty$

$$= \lim_{x \to \infty} \frac{e^x\left(1 - e^{-2x}\right)}{e^x\left(1 - e^{-2x}\right)}$$

$$= \lim_{x \to \infty} \frac{1 - e^{-2x}}{1 - e^{-2x}}$$

$$= \frac{1 - 0}{1 - 0}$$

$$= \boxed{1}$$

3. $\lim_{x \to -\infty} \dfrac{7}{8 + e^x} = \dfrac{7}{8 + 0}$

$$= \boxed{\frac{7}{8}}$$

4. $\lim_{x \to 0^+} \dfrac{\ln x + 5}{2 \ln x - 1} = \lim_{x \to 0^+} \dfrac{\ln x\left(1 + \frac{5}{\ln x}\right)}{\ln x\left(2 - \frac{1}{\ln x}\right)}$

$$= \lim_{x \to 0^+} \frac{1 + 0}{2 - 2}$$

$$= \boxed{\frac{1}{2}}$$

5. $\lim_{x\to\infty} \dfrac{\ln 5x^2}{\ln 2x} = \lim_{x\to\infty} \dfrac{\ln 5 + \ln x^2}{\ln 2 + \ln x}$

$= \lim_{x\to\infty} \dfrac{\ln 5 + 2\ln x}{\ln 2 + \ln x}$

$= \lim_{x\to\infty} \dfrac{\ln x\left(\frac{\ln 5}{\ln x} + 2\right)}{\ln x\left(\frac{\ln 2}{\ln x} + 1\right)}$

$= \dfrac{0+2}{0+1}$

$= \dfrac{2}{1}$

$= \boxed{2}$

COULD DO THIS WITH A SUBSTITUTION

LET $t = \ln x$.
AS $x \to \infty$, $t \to \infty$

$\lim_{x\to\infty} \dfrac{\ln 5 + 2\ln x}{\ln 2 + \ln x}$

$= \lim_{x\to\infty} \dfrac{\ln 5 + 2t}{\ln 2 + t}$

$= \lim_{x\to\infty} \dfrac{2t}{t}$

$= \boxed{2}$

EXTRA EXAMPLES

1. $\lim_{x \to +\infty} \frac{5 - x}{x + 7} = \lim_{x \to +\infty} \frac{-x}{x}$

$= \lim_{x \to +\infty} -1$

$= \boxed{-1}$

2. $\lim_{x \to -\infty} \frac{4x^3 + x^2 + 3x - 1}{7x^3 + 2x^2 - x + 3} = \lim_{x \to -\infty} \frac{4x^3}{7x^3}$

$= \lim_{x \to -\infty} \frac{4}{7}$

$= \boxed{\frac{4}{7}}$