MATH 121
L'HÔPITAL'S RULE

GOALS

- ☐ KNOW ALL OF THE INDETERMINATE FORMS (FOR WHEN L'HÔPITAL'S RULE APPLIES)
- ☐ USE L'HÔPITALS RULE

MAIN IDEA ALLOWS US TO WORK WITH LIMITS IN INDETERMINATE FORMS.

MAIN FORMS: $\frac{0}{0}$ AND $\frac{\infty}{\infty}$ ← L'HÔP APPLIES DIRECTLY

SECONDARY FORMS: $0\cdot\infty$, $\infty-\infty$, 1^{∞}, ∞^{0}, ... (NOT 1, THE CONSTANT; IT'S A FUNCTION THAT GOES TO 1; $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\to e$)

CAN BE REWRITTEN AS $\frac{0}{0}$, $\frac{\infty}{\infty}$

L'HÔPITAL FOR $\frac{0}{0}$ SUPPOSE f & g ARE CONTINUOUS AND DIFFERENTIABLE AT $x=a$, AND THAT $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

(1) $f(a)$ DEFINED
(2) $\lim_{x\to a} f(x)$ EXISTS
(3) $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(a)(x-a)+f(a)}{g'(a)(x-a)+g(a)}$$

SUB OUT FOR THE LOCAL LINEAR APPROXIMATIONS

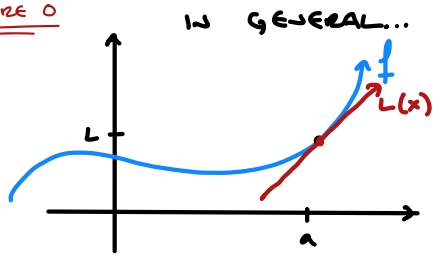
BOTH ARE 0

$$= \lim_{x\to 0}\frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}}$$

$$\boxed{\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}}$$

WHY IS THAT OK?

IN GENERAL...



$\lim_{x\to a} L(x) = \lim_{x\to a} f(x)$

e.g $\lim_{x\to 1} \frac{\ln(x)}{x-1}$ $\to \frac{\ln(1)=0}{1-1=0}$ $\overset{L'H}{=} \lim_{x\to 1} \frac{1/x}{1}$ $\to \frac{1}{1}$

$= \frac{1}{1}$

$= \boxed{1}$

L'HÔP ONLY WORKS FOR $\frac{0}{0}$ OR $\frac{\infty}{\infty}$:

$\lim_{x\to 1} \frac{\ln(x)}{x+1} = \frac{0}{2} = \boxed{0}$

~~BUT IF WE DO $\frac{f'}{g'}$: $\lim_{x\to 1} \frac{1/x}{1} = 1$~~

$\frac{A\to\infty}{B\to\infty} = \frac{\frac{1}{B}\to 0}{\frac{1}{A}\to 0}$

L'HÔP WORKS THE EXACT SAME WAY FOR $\frac{\infty}{\infty}$

$\frac{0}{0}$ IS "SAME" AS $\frac{\infty}{\infty}$

Ex $\lim_{x\to\infty} \frac{x\to\infty}{e^x\to\infty} \overset{L'H}{=} \lim_{x\to\infty} \frac{1\to 1}{e^x\to\infty}$

$= \boxed{0}$

Ex e^x WILL ALWAYS BEAT A POWER FUNCTION IN THE RACE TO ∞.

$$\lim_{x\to\infty} \frac{x^{100}}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{100\,x^{99}}{e^x}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{100\cdot 99\,x^{98}}{e^x}$$

$\vdots$

$100\cdot 99\cdot 98\cdot 97\cdots 3\cdot 2\cdot 1$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{100! \to \text{const}}{e^x \to \infty}$$

$= \boxed{0}$

NO NEED FOR L'HÔP

$\lim_{x\to\infty} \frac{2^x}{5^x} = \lim_{x\to\infty} \left(\frac{2}{5}\right)^x$

$= 0$

TRY ① $\lim_{x\to 0} \frac{\tan(3x)}{\ln(x+1)} \overset{L'H}{=} \lim_{x\to 0} \frac{3\sec^2 3x \to 3\cdot 1}{\frac{1}{x+1}\cdot 1 \to \frac{1}{1}}$

$= \frac{3}{1}$

$= \boxed{3}$

② $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x \to 0}{1 \to 1}$

$= \boxed{0}$

③ $\lim_{x\to\infty} \frac{3x^2+1}{5x^2+3x} \overset{L'H}{=} \lim_{x\to\infty} \frac{6x}{10x+3} \overset{L'H}{=} \lim_{x\to\infty} \frac{6}{10} = \boxed{\frac{3}{5}}$

RECAP

L'HÔPITAL TELLS US THAT $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ PROVIDED THAT...

- f AND g ARE DIFFERENTIABLE
- $\frac{f}{g} \to \boxed{\frac{0}{0} \text{ OR } \frac{\infty}{\infty}}$ ← ONLY INDET FORMS FOR WHICH LHÔP WORKS DIRECTLY

TRY:

① $\lim_{x\to 1} \frac{x^9-1}{x^5-1}$ ② $\lim_{x\to\infty} \frac{\ln(x^2)}{\sqrt{x}}$ ③ $\lim_{x\to 0} \frac{\arctan x}{x}$

④ $\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos(\pi x)}$

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① $\lim_{x\to 1} \frac{x^9-1}{x^5-1} \overset{L'H}{=} \lim_{x\to 1} \frac{9x^8}{5x^4}$ ($0/0$)

$x^9-1 = (x-1)(x^8+\cdots+1)$

$= \boxed{\frac{9}{5}}$

② $\lim_{x\to\infty} \frac{\ln(x^2)}{\sqrt{x}} = \lim_{x\to\infty} \frac{2\ln x}{x^{1/2}}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{2/x \to 0}{\frac{1}{2}x^{-1/2} \to 0}$ DON'T DO L'HÔP! SIMPLIFY.

$= \lim_{x\to\infty} \frac{4\sqrt{x}}{x}$ ← $\frac{x^{1/2}}{x^1} = \frac{1}{x^{1-1/2}}$

$$= \lim_{x\to\infty} \frac{4}{\sqrt{x}} \quad (4 \to 4,\ \sqrt{x} \to \infty)$$

$$= \boxed{0}$$

$\lim_{x\to\infty} x = \infty \to \sqrt{\lim_{x\to\infty} x} = \infty$

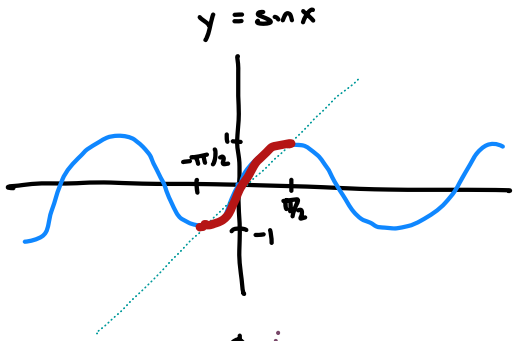
$\lim_{x\to\infty} \sqrt{x} = \infty$

③ $\lim_{x\to 0} \frac{\arctan x}{x}$ ($\arctan x \to 0$, $x \to 0$)

$$= \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1}$$

$$= \boxed{1}$$

$y = \arcsin x$ IS A FUNCTION



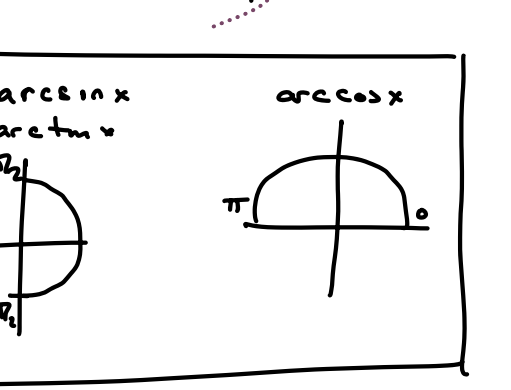
④ $\lim_{x\to 1} \frac{1 - x + \ln x}{1 + \cos(\pi x)}$ ($1 - x + \ln x \to 0$, $1 + \cos(\pi x) \to 0$)

$$\overset{L'H}{=} \lim_{x\to 1} \frac{-1 + \frac{1}{x}}{-\pi \sin(\pi x)}$$

$$\overset{L'H}{=} \lim_{x\to 1} \frac{-\frac{1}{x^2}}{-\pi^2 \cos(\pi x)}$$

$$= \frac{-1}{-\pi^2(-1)}$$

$$= \boxed{-\frac{1}{\pi^2}}$$



OTHER INDET. FORMS: REWRITE AS OR RELATE TO $\frac{0}{0}$ OR $\frac{\infty}{\infty}$.

0·∞: $\lim_{x\to\infty} x e^{-2x}$ ($x \to \infty$, $e^{2x} \to \infty$, $0 \cdot \infty$)

$$= \lim_{x\to\infty} \frac{x}{e^{2x}}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{2e^{2x}}$$

$$= \boxed{0}$$

$0 \cdot \infty$ ISN'T ALWAYS 0: $\lim_{x \to 0^+} x \csc x \longrightarrow 0 \cdot \infty = \lim_{x \to 0^+} \frac{x}{\sin x} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$

$\overset{L'H}{=} \lim_{x \to 0} \frac{1}{\cos x}$

$= \boxed{1}$

1^∞:

$$\lim_{x \to 0^+} (1+2x)^{5/x} \longrightarrow 1^\infty$$

* USE THE FACT THAT $\ln A = B \longleftrightarrow A = e^B$ *

- FIRST COMPUTE $\underbrace{\lim \ln (1+2x)^{5/x}}_{\ln A}$
- CALL ANS B, SO $\ln A = B$
- FINAL ANS. $A = e^B$

$$\lim_{x \to 0^+} \ln (1+2x)^{5/x} = \lim_{x \to 0^+} \frac{5}{x} \ln (1+2x) \longrightarrow \infty \cdot 0$$

$$= \lim_{x \to 0^+} \frac{5 \ln (1+2x)}{x} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$$ L'HÔP TIME

$$= \lim_{x \to 0^+} \frac{5 \cdot \frac{2}{1+2x}}{1}$$

$$= 5 \cdot \frac{2}{1}$$

$$= (10) \longleftarrow \text{NOT FINAL ANS.}$$

$$\boxed{\lim_{x \to 0^+} (1+2x)^{5/x} = e^{10}}$$