MATH 122

INTEGRATION BY PARTS

GOALS

- ☐ KNOW WHEN/HOW TO USE INTEGRATION BY PARTS
- ☐ USE LIATED TO DETERMINE CHOICES FOR u, dv
- ☐ USE I.B.P. W/ SUBSTITUTION

GOAL: INTEGRATE THINGS LIKE...

$$\int xe^x dx, \quad \int x\cos x\,dx, \quad \int \ln x\,dx, \quad \int \arctan x\,dx, \quad \int e^x \cos x\,dx, \quad \int \frac{\ln x}{x^5}\,dx, \ldots$$

PRODUCT RULE:

$$(uv)' = uv' + u'v$$

$$\int (uv)' = \int uv' + \int u'v$$

$$uv = \boxed{\int uv'} + \int u'v \qquad \text{SOLVE FOR ME.}$$

$$\int uv' = uv - \int u'v$$

W/ FORMAL NOTATION: $\boxed{\int u\,dv = uv - \int v\,du.}$

Ex: $\int x e^x dx$ CHOOSE u AND dv TO FIT $\int u\,dv$.

$u = x$ $\quad dv = e^x dx$

$du = dx$ $\quad \int dv = \int e^x dx$

$v = e^x$

$$\int u\,dv = uv - \int v\,du.$$

$$\int x e^x dx = x e^x - \int e^x dx$$

$$= \boxed{x e^x - e^x + C}$$

check:

$(xe^x - e^x) = (1)e^x + xe^x - e^x$

$= \boxed{xe^x}$ ✓

How to choose: DETAIL SPELLED BACKWARDS . . .

u comes first, dv is the rest

- L — LOGS
- I — INVERSE (TRIG)
- A — ALGEBRAIC $(x, 3x, x^2, \frac{1}{x}, \sqrt{x}, \ldots)$
- T — TRIG.
- E — EXPONENTIAL
- D — DIFFERENTIAL (dx)

$\int x e^x dx$ (u = x, dv = $e^x dx$)

× × ✓

L I A T E D

$\int x^3 \ln x\, dx$ (u = $\ln x$)

✓

L I A T E D

$u = \ln x$ $\quad dv = x^3 dx$

$\int \arctan x\, dx$

×

L I A T E D

$u = \arctan x$

$dv = dx$

$$\int u\,dv = uv - \int v\,du.$$ SIMPLIFY

Ex: $\int x^3 \ln x\, dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}dx$

$\vdots \quad - \int \frac{1}{4}x^3 dx$

$$\boxed{\frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C}$$

$u = \ln x$ $\quad dv = x^3 dx$

↓ DERIV. $\quad$ ↓ ANTI DERIV.

$du = \frac{1}{x}dx$ $\quad v = \frac{x^4}{4}$

Ex: $\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x$

SUBSTITUTE

$$= \boxed{x\arctan x - \tfrac{1}{2}\ln|1+x^2| + C}$$

LET $t = 1 + x^2$

$dt = 2x\,dx$

$\frac{1}{2}\,dt = x\,dx$

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \tfrac{1}{2}\ln|t|$$

$$= \tfrac{1}{2}\ln|1+x^2|$$

CHECK: $(1)\arctan x + x\cdot\frac{1}{1+x^2} - \frac{1}{2}\cdot 2x\cdot\frac{1}{1+x^2}$

$\arctan x$ ✓

Ex:

$$\int e^x \cos x\,dx = e^x\cos x - \int -e^x \sin x\,dx$$

$$= e^x\cos x + \int e^x \sin x\,dx$$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x\,dx \qquad v = e^x$

$u = \sin x \qquad dv = e^x dx$

$du = \cos x\,dx \qquad v = e^x$

$$\int e^x\cos x\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx$$

$$+\int e^x\cos x\,dx \qquad +\int e^x\cos x\,dx$$

$$2\int e^x\cos x\,dx = e^x\cos x + e^x\sin x$$

$A = B + C - A$

$2A = B + C$

$A = \frac{1}{2}B + \frac{1}{2}C$

$$\int e^x\cos x\,dx = \boxed{\tfrac{1}{2}e^x\cos x + \tfrac{1}{2}e^x\sin x + C}$$

Ex:

① $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$

$= \boxed{x\sin x + \cos x + C}$

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

② $\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx$

$\vdots \quad - \int dx$

$= \boxed{x\ln x - x + C}$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\,dx \qquad v = x$

Ex: $\int e^{\sqrt{x}}\,dx$

Let $\boxed{t = \sqrt{x}} \longrightarrow dt = \frac{1}{2\sqrt{x}}\,dx$

$= \frac{1}{2t}\,dx$

$\boxed{2t\,dt = dx}$

$\int 2te^t\,dt = 2te^t - \int 2e^t\,dt$

$= 2te^t - 2e^t + C$

$= \boxed{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}$

$u = 2t \qquad dv = e^t dt$

$du = 2dt \qquad v = e^t$

Ex: $\int_{\pi/4}^{\pi/2} 3x\csc^2 x\,dx$

$u = 3x$	$dv = \csc^2 x\,dx$
$du = 3dx$	$v = -\cot x$

$$\int 3x\csc^2 x\,dx = -3x\cot x + \int 3\cot x\,dx$$

$$\int_{\pi/4}^{\pi/2} 3x\csc^2 x\,dx = -3x\cot x + 3\ln|\sin x| \Big|_{\pi/4}^{\pi/2}$$

$$= 0 + 0 - \left(-\frac{3\pi}{4}(1) + 3\ln\left(\frac{\sqrt{2}}{2}\right)\right)$$

$$= \boxed{\frac{3\pi}{4} - 3\ln\left(\frac{\sqrt{2}}{2}\right)}$$

$$\int x f''(x)\,dx = x f'(x) - \int f'(x)\,dx \qquad u = x \qquad dv = f''(x)\,dx$$

$$= x f'(x) - f(x)\Big|_0^1 \qquad du = 1\,dx \qquad v = f'(x)$$

$$= [1(f'(1)) - f(1)] - [0 - f(0)]$$