MATH 122

INTEGRATING TRIG FUNCTIONS

GOALS

- ☐ INTEGRATE POWERS OF SINE AND COSINE
- ☐ KNOW WHEN AND HOW TO USE TRIG IDENTITIES

INTEGRATING TRIG FUNCTIONS

Ex: $\int \sin^2 x \cos^3 x \, dx = \int \overbrace{\sin^2 x}^{u^2} \overbrace{\cos^2 x}^{\text{WRITE AS SINES}} \overbrace{\cos x \, dx}^{du} \longrightarrow u = \sin x$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

$$= \int u^2 (1 - u^2) \, du.$$

$$= \int u^2 - u^4 \, du.$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$

$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$

$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$

WHY IT DOESNT WORK THE OTHER WAY:

$$\int \sin^2 x \cos^3 x \, dx = \int \underbrace{\sin x}_{?} \underbrace{\cos^3 x}_{u^3} \underbrace{\sin x \, dx}_{-du.}$$

$u = \cos x$

$du = -\sin x \, dx$

$\sqrt{1 - \cos^2 x}$ EW!!

$$= -\int \sqrt{1 - u^2} \, u^3 \, du.$$

Ex: $\int \sin^5 x \cos^4 x \, dx = \int \underbrace{\sin^4 x}_{(\sin^2 x)^2} \cos^4 x \sin x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$= \int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx$

$= -\int (1 - u^2)^2 u^4 \, du.$

$= -\int (1 - 2u^2 + u^4) u^4 \, du.$

$= -\int u^4 - 2u^6 + u^8 \, du.$

$= -\left(\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} \right) + C$

$= \boxed{-\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + C}$

Ex: $\int \sin^3 x \cos^3 x \, dx = \int \underbrace{\sin^3 x}_{u^3} \underbrace{\cos^2 x}_{(1-\sin^2 x)} \overbrace{\cos x \, dx}^{du.}$

$u = \sin x$

$du = \cos x \, dx$

$= \int u^3 (1 - u^2) \, du.$

$= \int u^3 - u^5 \, du.$

$= \frac{u^4}{4} - \frac{u^6}{6} + C$

$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C \quad \text{or} \quad \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$

Ex:

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1-\cos 2x)\cdot\frac{1}{2}(1+\cos 2x)\,dx$$

$$= \frac{1}{4}\int 1 - \cos^2 2x \, dx$$

$$= \frac{1}{4}\int 1 - \frac{1}{2}(1+\cos 4x)\,dx$$

$$= \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x \, dx$$

$$= \frac{1}{4}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x\right) + C$$

$$= \boxed{\frac{1}{8}x - \frac{1}{32}\sin 4x + C}$$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin A\cos B = \frac{1}{2}(\sin(A-B)+\sin(A+B))$

$\cos A\cos B = \frac{1}{2}(\cos(A-B)+\cos(A+B))$

$\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$

SUMMARY: $\int \sin^m x \cos^n x \, dx$

- IF m IS ODD: PULL ASIDE ONE $\sin x$ AND LET $u = \cos x$
- IF n IS ODD: PULL ASIDE ONE $\cos x$ AND LET $u = \sin x$
- IF m & n ARE EVEN: USE DOUBLE ANGLE.

Ex:

$$\int \sin(\overset{A}{4x})\sin(\overset{B}{3x})\,dx = \frac{1}{2}\int \cos(4x-3x) - \cos(4x+3x)\,dx$$

$$= \frac{1}{2}\int \cos x - \cos 7x \, dx$$

$$= \boxed{\frac{1}{2}\left(\sin x - \frac{1}{7}\sin 7x\right) + C}$$

$\sin A\cos B = \frac{1}{2}(\sin(A-B)+\sin(A+B))$

$\cos A\cos B = \frac{1}{2}(\cos(A-B)+\cos(A+B))$

$\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$

TANGENT & SECANT:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \qquad u = \cos x$$

$$du = -\sin x\,dx$$

$$= -\int \frac{1}{u}\,du.$$

$$= \boxed{-\ln|\cos x| + C} \quad \text{OR} \quad \boxed{\ln|\sec x| + C}$$

$$\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx \qquad u = \sec x + \tan x$$

$$du = \sec x \tan x + \sec^2 x\,dx$$

$$= \int \frac{1}{u}\,du.$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$