MATH 122

IMPROPER INTEGRALS

GOALS

- ☐ IDENTIFY TWO TYPES OF IMPROPER INTEGRALS
- ☐ REWRITE IMPROPER INTEGRALS AS LIMITS

TWO TYPES:

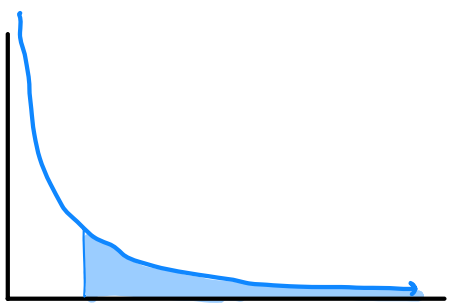
① BOUNDS CONTAIN $\pm\infty$

e.g. $\int_1^\infty \frac{1}{x^2}\,dx$ $\qquad$ $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

② INTERVAL CONTAINS AN INFINITE DISCONTINUITY

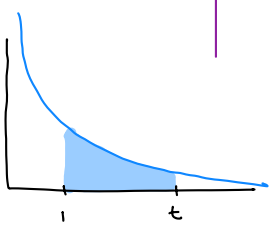
e.g. $\int_0^2 \frac{1}{(x-2)^3}\,dx$ $\qquad$ $\int_0^4 \frac{1}{\sqrt{x}}\,dx$

ex: $\int_1^\infty \frac{1}{x^2}\,dx$



TREAT IT LIKE A LIMIT USING A PLACEHOLDER VARIABLE FOR ∞.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$



$$= \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t$$

$$= \lim_{t\to\infty} -\frac{1}{t} - \frac{-1}{1} = 0 + 1 = \boxed{1}$$

IT'S POSSIBLE FOR A SUM OF INFINITE TERMS (≥ 0) TO YIELD A FINITE VALUE

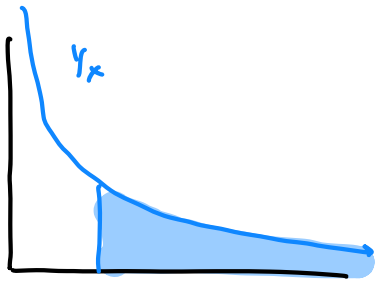
$0.\overline{9} = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$

$0.\overline{9} = 1$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$



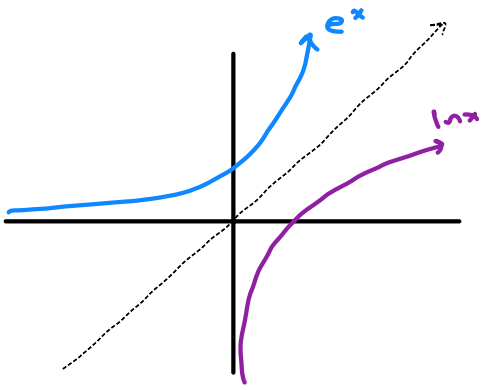
Ex: $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$



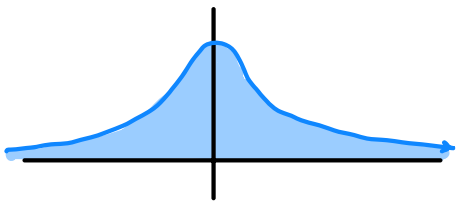
$= \lim_{t\to\infty} \ln|x| \Big|_1^t$

$= \lim_{t\to\infty} \ln|t| - \ln|1|$

$= \boxed{\infty}$ DIVERGES



Ex: $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

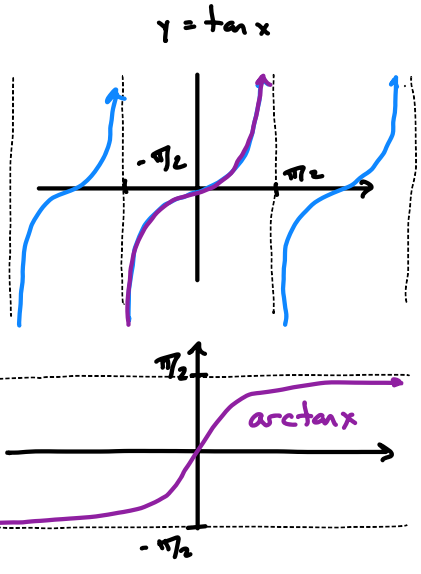


$\lim_{t\to\infty} \int_{-t}^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \arctan x \Big|_{-t}^{t}$

$= \lim_{t\to\infty} (\arctan t - \arctan(-t))$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$

$= \boxed{\pi}$



Ex: $\int_0^\infty x e^{-x}\,dx$

GET ANTIDERIV.

$$\int x e^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx \qquad u = x \qquad dv = e^{-x}\,dx$$

$$du = dx \qquad v = -e^{-x}$$

$$= -xe^{-x} - e^{-x} + C$$

$$\int_0^\infty x e^{-x}\,dx = \lim_{t\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \left(-te^{-t} - e^{-t}\right) - (0 - 1)$$

$-te^{-t} \to 0$

L'HÔPITAL!

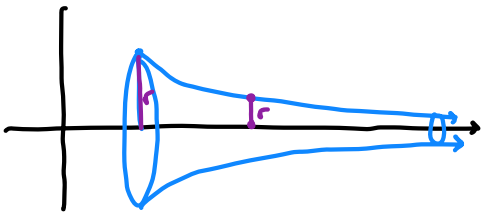
$$\lim_{t\to\infty} -\frac{t}{e^t} \quad \left(\frac{t\to\infty}{e^t\to\infty}\right)$$

$$= \lim_{t\to\infty} \frac{-1}{e^t}$$

$$= 0$$

$$= \boxed{1}$$

Ex: REVOLVE REGION ENCLOSED BY $y = 1/x$, $y = 0$, $x = 1$ ABOUT X-AXIS.



$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx$$

$$= \pi \int_1^\infty \frac{1}{x^2}\,dx$$

DID THIS EARLIER.

$$= \pi(1)$$

$$= \boxed{\pi}$$

Ex: $\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^4 \frac{1}{\sqrt{x}}\,dx$

0 — VERT. ASY

$= \lim_{t\to 0^+} 2\sqrt{x}\,\Big|_t^4$

$= \lim_{t\to 0^+} 2\sqrt{4} - 2\sqrt{t}$

$= 2\cdot 2 - 2\cdot 0$

$= \boxed{4}$

