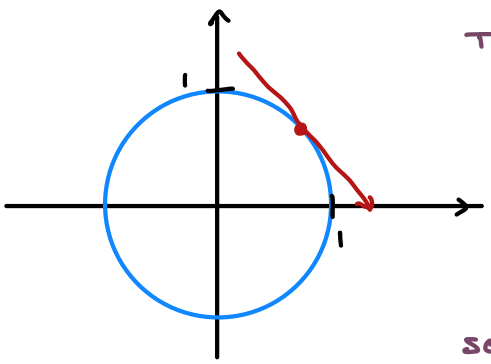
MATH 121
IMPLICIT DIFFERENTIATION

GOALS

- ☐ PERFORM IMPLICIT DIFFERENTIATION USING ALL OF OUR DERIVATIVE RULES
- ☐ COMPUTE $\frac{d^2y}{dx^2}$
- ☐ USE IMPLICIT DIFFERENTIATION TO FIND AN EQUATION FOR THE LINE TANGENT TO A CURVE AT A GIVEN POINT

* "IMPLICIT" REFERS TO THE FACT THAT WE'LL TREAT y LIKE A FUNCTION OF x, EVEN IF IT ISN'T.

e.g. $x^2 + y^2 = 1$
UNIT CIRCLE.



TAKE DERIVATIVE OF BOTH SIDES w/ RESPECT TO x:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1)$$

$$2x + 2yy' = 0$$

SOLVE FOR y':

$$2yy' = -2x$$

$$\boxed{y' = -\frac{x}{y}}$$

TREAT y LIKE IT'S A FUNCTION OF x:

$$\frac{d}{dx}\left((y)^2\right)$$
$$= 2(y)^1 y'$$
$$= \boxed{2yy'}$$

CHAIN RULE!

$$\boxed{y - \frac{\sqrt{2}}{2} = -\left(x - \frac{\sqrt{2}}{2}\right)}$$

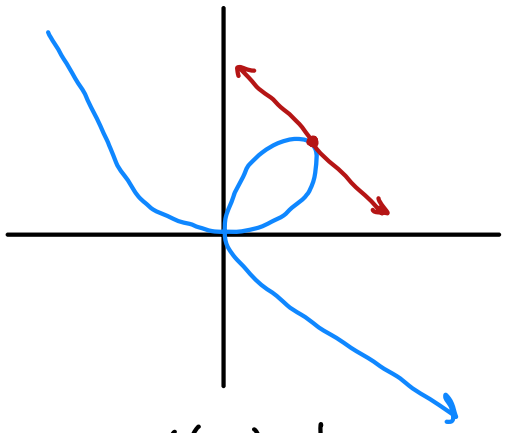
WE USE THIS LIKE ANY OTHER DERIVATIVE: FIND TANGENT LINE AT $(-\sqrt{2}/2, \sqrt{2}/2)$

$$y'\left(\sqrt{2}/2, \sqrt{2}/2\right) = -\frac{\sqrt{2}/2}{\sqrt{2}/2} = -1$$ SLOPE

Ex: $x^3 + y^3 - 2xy = 0$ ← CAN'T SOLVE FOR y AT ALL.

RENÉ

"FOLIUM OF DESCARTES"



FIND TANGENT AT $(1,1)$:

$y' = \frac{dy}{dx}$

$$\frac{d}{dx}\left(x^3 + y^3 - \overset{A\ B}{2xy}\right) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2 y' - \underset{A'\ B \quad A\ B'}{\left(2(1)y + 2xy'\right)} = 0$$

$$3x^2 + 3y^2 y' - 2y - 2xy' = 0$$

$$3y^2 y' - 2xy' = 2y - 3x^2$$

$$y'(3y^2 - 2x) = 2y - 3x^2$$

$$\boxed{y' = \frac{2y - 3x^2}{3y^2 - 2x}}$$

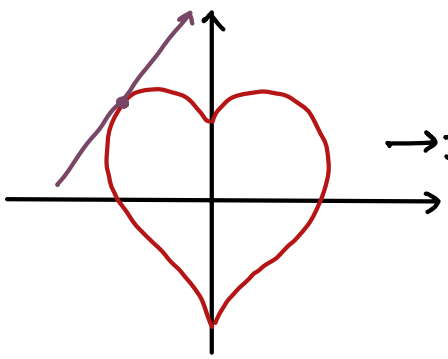
$m = y'(1,1)$

$= \frac{2-3}{3-2}$

$= \boxed{-1}$

$y - 1 = -(x-1)$

Ex: $(x^2 + y^2 - 1)^3 - x^2 y^3 = 0$; $P(-1,1)$



$$\frac{d}{dx}\left((x^2+y^2-1)^3 - x^2y^3 = 0\right)$$

$$\rightarrow 3(x^2+y^2-1)^2(2x + 2yy') - \left(2xy^3 + x^2(3y^2y')\right) = 0$$

PLUG IN $(-1,1)$ FIRST:

$$3(1+1-1)^2(-2 + 2y') - (-2 + 3y') = 0$$

$$-6 + 6y' + 2 - 3y' = 0$$

SOLVE FOR y':

$$3y' = 4$$

$$\boxed{y' = 4/3}$$

$y - 1 = \frac{4}{3}(x+1)$

Ex: $x^4 - x^2y^3 + y^4 = 1 \quad ; \quad P(-1,1)$

- FIND y'
- USE y' TO GET AN EQUATION FOR THE TANGENT AT P.

$$x^4 - \overset{A}{x^2}\overset{B}{y^3} + y^4 = 1 \qquad \text{(CHAIN)}$$

$$4x^3 - \left(\underset{A'}{2x}\underset{B}{y^3} + \underset{A}{x^2}\left(\underset{B'}{3y^2y'}\right)\right) + 4y^3y' = 0$$

$$4x^3 - 2xy^3 - 3x^2y^2y' + 4y^3y' = 0$$

$$y'(-3x^2y^2 + 4y^3) = 2xy^3 - 4x^3$$

$$\boxed{y' = \frac{2xy^3 - 4x^3}{4y^3 - 3x^2y^2}} \quad \text{OR} \quad \boxed{y' = \frac{4x^3 - 2xy^3}{3x^2y^2 - 4y^3}}$$

Ex: $x^2\cos y + x^3\sin(2y) = 1 \qquad P(1,0)$

- FIND TANGENT LINE (DON'T NEED TO SOLVE FOR y' FIRST UNLESS YOU WANT TO).

$$\underset{A}{x^2}\underset{B}{\cos y} + \underset{C}{x^3}\underset{D}{\sin(2y)} = 1 \qquad \text{(CHAIN RULE)}$$

$$2x\cos y + x^2(-\sin(y)\cdot y') + 3x^2\sin(2y) + x^3\cos(2y)\cdot 2y' = 0$$

PLUG IN $x = 1 \quad y = 0$

$$2(1) + 1(0) + 0 + 2y' = 0$$

$$2y' = -2$$

$$\boxed{y' = -1}$$

$$y - 0 = -(x - 1)$$

$$\boxed{y = -x + 1}$$

COMPUTING $\frac{d^2y}{dx^2}$

* SINCE $\frac{dy}{dx}$ IS IN TERMS OF x AND y, WE CAN WRITE $\frac{d^2y}{dx}$ IN TERMS OF x AND y, TOO.

EX: $x^2 - xy + y^2 = 1$

$$2x - y - xy' + 2yy' = 0$$

$$y' = \frac{y - 2x}{2y - x}$$

$$y'' = \frac{(2y-x)(y'-2) - (2y'-1)(y-2x)}{(2y-x)^2}$$

$$= \frac{2yy' - 4y - xy' + 2x - 2yy' + 4xy' + y - 2x}{(2y-x)^2}$$

$$= \frac{3xy' - 3y}{(2y-x)^2}$$

$$= \frac{3x\left(\frac{y-2x}{2y-x}\right) - 3y}{(2y-x)^2}$$

$$= \frac{3x(y-2x) - 3y(2y-x)}{(2y-x)^3}$$

$$= \frac{3xy - 6x^2 - 6y^2 + 3xy}{(2y-x)^3}$$

$$= \frac{6xy - 6x^2 - 6y^2}{(2y-x)^3}$$