MATH 121

DIFFERENTIATION TECHNIQUES

GOALS

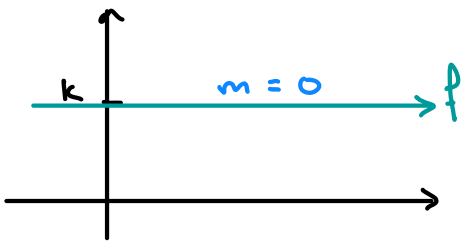
- USE THE LIMIT DEFINITION OF THE DERIVATIVE TO FIND "SHORTCUTS" FOR COMPUTING THE DERIVATIVES OF CONSTANTS, POWER FUNCTIONS, AND TRIG FUNCTIONS
- USE THE LIMIT DEFINITION OF THE DERIVATIVE TO COME UP WITH RULES FOR COMBINING DERIVATIVES
  - SUMMATION RULE, CONSTANT MULTIPLE RULE, PRODUCT RULE, AND QUOTIENT RULE

NOTATION $\frac{d}{dx}$ MEANS "THE DERIVATIVE OF ..." (INCOMPLETE PHRASE)

e.g. $\frac{d}{dx}\left(2\sqrt{x}\right) = \frac{1}{\sqrt{x}}$

"THE DERIVATIVE OF $2\sqrt{x}$ IS $\frac{1}{\sqrt{x}}$"

CONSTANT FUNCTIONS $f(x) = k$



$$\frac{d}{dx}(k) = 0$$

PROOF: $f(x+h) = k$

$$f'(x) = \lim_{h \to 0} \frac{k - k}{h}$$

$$= \boxed{0}$$

e.g. $g(x) = 7 \longrightarrow g'(x) = 0$

$h(x) = 13\pi^3 \longrightarrow h'(x) = 0$

$i(x) = e^{23} \longrightarrow i'(x) = 0$

POWER FUNCTIONS: $f(x) = x^n$ (n IS A POSITIVE INTEGER)

SO e^x IS NOT A POWER FUNCTION!

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

AT LEAST TWO h's IN EACH TERM.

$$= \lim_{h\to 0} \frac{(x^n + nx^{n-1}h + \cdots + h^n) - x^n}{h}$$

$$= \lim_{h\to 0} \frac{nx^{n-1}h + \cdots + h^n}{h}$$

AT LEAST ONE h.

$$= \lim_{h\to 0} (nx^{n-1} + \cdots + h^{n-1})$$

$$= \boxed{nx^{n-1}}$$

PASCAL'S TRIANGLE:

1	$(x+h)^0 = 1$
1 1	$(x+h)^1 = x+h$
1 2 1	$(x+h)^2 = x^2 + 2xh + h^2$
1 3 3 1	$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$
1 4 6 4 1	$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$
1 5 10 10 5 1	

NOT NEEDED: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

e.g $g(x) = x^6 \longrightarrow g'(x) = 6x^{6-1} = \boxed{6x^5}$

$h(x) = \sqrt[3]{x} = x^{1/3} \longrightarrow h'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3} = \boxed{\frac{1}{3\sqrt[3]{x^2}}}$

$i(x) = \frac{1}{x^4} = x^{-4} \longrightarrow i'(x) = -4x^{-4-1} = -4x^{-5} = \boxed{\frac{-4}{x^5}}$

MUST BE IN x^n FORM BEFORE WE USE POWER RULE

COMBINING DERIVATIVES:

CONSTANT MULTIPLE

$$\frac{d}{dx}\left(k f(x)\right) = k f'(x)$$

e.g. $y = 3x^7$

$y' = 3(7x^6)$

$= 21x^6$

FACTOR OUT.

$$\lim_{h} \frac{k f(x+h) - k f(x)}{h}$$

$$k \lim_{h} \frac{f(x+h) - f(x)}{h}$$

$$k \cdot f'(x)$$

SUM / DIFFERENCE

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$$

e.g. $h(x) = 3x^7 + 2\sqrt{x}$

$h'(x) = 21x^6 + \frac{1}{\sqrt{x}}$

$$\lim_{h} \frac{f(x+h) + g(x+h) - \left(f(x) + g(x)\right)}{h}$$

$$\lim_{h} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

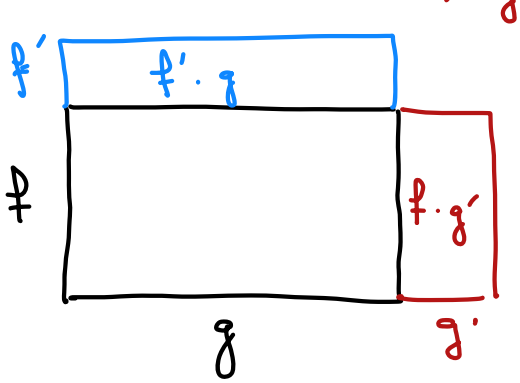
$$f'(x) + g'(x)$$

PRODUCT RULE

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

$$\lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

CAN'T BE WRITTEN AS $f' \cdot g'$.



*NOT REALLY A PROOF...

QUOTIENT RULE

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2}$$

SHORT HAND:

$$\left(\frac{a}{b}\right)' = \frac{ba' - b'a}{b^2}$$

e.g. $y = \dfrac{4x+1}{x^2+3}$ ← a, ← b

$$y' = \frac{(x^2+3)(4) - (2x)(4x+1)}{(x^2+3)^2}$$

$$= \frac{4x^2 + 12 - 8x^2 - 2x}{(x^2+3)^2}$$

$$= \boxed{\frac{12 - 2x - 4x^2}{(x^2+3)^2}}$$

$a = 4x+1$

$a' = 4(1x^0) + 0$

$a' = 4$

$b = x^2 + 3$

$b' = 2x^1 + 0$

$b' = 2x$

SUMMARY

CONSTANT RULE

$$\frac{d}{dx}(k) = 0$$

POWER RULE

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

CONST. MULTIPLE RULE

$$\frac{d}{dx}(kf(x)) = kf'(x)$$

SUM/DIFFERENCE RULE

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$$

PRODUCT RULE

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

QUOTIENT RULE

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - g'(x)f(x)}{g(x)^2}$$

TRY: COMPUTE THE DERIVATIVE.

① $f(x) = 2x^5 - x^3 + 4x - 1$

$$\boxed{f'(x) = 10x^4 - 3x^2 + 4}$$

② $g(x) = \dfrac{4}{\sqrt{x}}$

QUOTIENT:

$g(x) = \dfrac{4}{x^{1/2}}$ (a / b)

$g'(x) = \dfrac{x^{1/2}(0) - \frac{1}{2}x^{-1/2} \cdot 4}{(x^{1/2})^2}$ (b, a′, b′, a, b^2)

$= \dfrac{-2x^{-1/2}}{x}$

$= \dfrac{-2}{x^{3/2}}$

FINE TOO: $\boxed{-2x^{-3/2}}$

$= \dfrac{-2}{\sqrt{x^3}}$

SIMPLIFY FIRST:

$g(x) = 4\dfrac{1}{x^{1/2}}$

$= 4x^{-1/2}$

$g'(x) = 4\left(-\frac{1}{2}x^{-3/2}\right)$

$= -2x^{-3/2}$

$= \boxed{\dfrac{-2}{\sqrt{x^3}}}$

③ $h(x) = \dfrac{2x^2 - x + 7}{x}$ (a / b)

QUOTIENT:

$h'(x) = \dfrac{x(4x-1) - (1)(2x^2 - x + 7)}{x^2}$

$= \dfrac{4x^2 - x - 2x^2 + x - 7}{x^2}$

$= \boxed{\dfrac{2x^2 - 7}{x^2}}$ → $\dfrac{2x^2}{x^2} - \dfrac{7}{x^2}$

$2 - \dfrac{7}{x^2}$

¡SAME!

SIMPLIFY FIRST:

$h(x) = \dfrac{2x^2}{x} - \dfrac{x}{x} + \dfrac{7}{x}$

$= 2x - 1 + 7x^{-1}$

$h'(x) = 2 - 7x^{-2}$

$= \boxed{2 - \dfrac{7}{x^2}}$

TRIG

$f(x) = \sin x$ $\qquad f'(x) = \lim_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h \to 0} \frac{\sin x \cosh + \sinh \cos x - \sin x}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sin x \cosh - \sin x}{h} + \frac{\sinh \cos x}{h} \right)$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cosh - 1}{h} + \cos x \cdot \frac{\sinh}{h} \right)$$

($\cosh \to 0$, $\sinh \to 1$)

$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$= \boxed{\cos x}$$

MAIN TWO:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Ex: $y = \underbrace{x^2}_{A} \underbrace{\sin x}_{B}$ $\qquad y' = AB' + A'B$

$$\boxed{y' = x^2 \cos x + 2x \sin x}$$

OTHER FOUR

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \quad \begin{matrix} a \\ b \end{matrix}$$

$$= \frac{\overset{b}{\cos x}\,\overset{a'}{(\cos x)} - \overset{b'}{(-\sin x)}\,\overset{a}{(\sin x)}}{\underset{b^2}{\cos^2 x}}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \boxed{\sec^2 x}$$

NOTE ON NOTATION:

$\frac{d}{dx}$ MEANS "DERIV. OF..."
↑ OPERATION LIKE $+, -, \div$,

$\frac{dy}{dx}$ MEANS DERIV. OF y (W/ RESPECT TO x)

e.g. $\boxed{\frac{d}{dx}(x^2) = 2x}$ VERSUS $\boxed{y = x^2 \Rightarrow \frac{dy}{dx} = 2x}$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{\cos x \cdot 0 - (-\sin x)\cdot 1}{\cos^2 x}$$

$$= \boxed{\frac{\sin x}{\cos^2 x}}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \boxed{\sec x \tan x}$$

TRIG DERIVATIVES

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc^2 x \cot^2 x$$

* ALL "CO-" FUNCTIONS HAVE - SIGN *

TRY: FOR 1-3, COMPUTE THE DERIVATIVE

① $f(x) = x^3 \sin x$ ② $g(x) = \dfrac{x+1}{\tan x}$ ③ $h(x) = \sin x \cos x$

④ FIND AN EQUATION FOR THE TANGENT LINE TO $y = \tan x$ AT $x = \frac{3\pi}{4}$

⑤ FIND ALL x-VALUES WHERE $y = 2\sin x - x$ HAS A HORIZONTAL TANGENT LINE.

① $f(x) = \underset{A}{x^3} \underset{B}{\sin x}$

$$f'(x) = \underset{A' \cdot B}{3x^2 \sin x} + \underset{A \cdot B'}{x^3 \cos x}$$

② $g(x) = \dfrac{x+1}{\tan x}$

SIMPLIFY:

$$g(x) = (x+1)\cot x$$

$$g'(x) = (1)\cot x + (x+1)(-\csc^2 x)$$

$$= \boxed{\cot x - (x+1)\csc^2 x}$$

QUOTIENT:

$$g'(x) = \boxed{\frac{\tan x (1) - \sec^2 x (x+1)}{\tan^2 x}}$$

$$= \frac{\tan x}{\tan^2 x} \cdot \frac{\sec^2 x}{\tan^2 x}(x+1)$$

$$= \frac{1}{\tan x} - \frac{1}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x}(x+1)$$

$$= \boxed{\cot x - \csc^2 x (x+1)}$$

③ $h(x) = \sin x \cos x$

$$h'(x) = \cos x \cos x + \sin x (-\sin x)$$

$$= \boxed{\cos^2 x - \sin^2 x}$$

④ FIND AN EQUATION FOR THE TANGENT LINE TO $y = \tan x$ AT $x = \frac{3\pi}{4}$

POINT: $x = \frac{3\pi}{4}$

$$y\left(\frac{3\pi}{4}\right) = \tan\left(\frac{3\pi}{4}\right)$$

$$y = \boxed{-1}$$

SLOPE: $y' = \sec^2 x$

$$y'\left(\frac{3\pi}{4}\right) = \sec^2\left(\frac{3\pi}{4}\right)$$

$$= \left(\sec\left(\frac{3\pi}{4}\right)\right)^2$$

$$= \left(-\sqrt{2}\right)^2$$

$$m = 2$$

LINE: $y - (-1) = 2\left(x - \frac{3\pi}{4}\right)$

$$\boxed{y = 2\left(x - \frac{3\pi}{4}\right) - 1}$$

⑤ FIND ALL X-VALUES WHERE $y = 2\sin x - x$ HAS A HORIZONTAL TANGENT LINE.

$y' = 0$

$$y' = 2\cos x - 1 = 0$$

$$2\cos x = 1$$

$$\cos x = 1/2$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{-\pi}{3}, \ldots$$

$$x = \frac{\pi}{3} + 2\pi n \quad \text{AND} \quad \frac{5\pi}{3} + 2\pi n$$