MATH 121

THE DEFINITION OF THE DERIVATIVE

GOALS

- ☐ DEFINE THE DERIVATIVE USING LIMITS
- ☐ APPLY THE LIMIT DEFINITION OF THE DERIVATIVE TO COMPUTE f' FOR QUADRATIC FUNCTIONS, SQUARE ROOT FUNCTIONS, $1/x$, AND MORE
- ☐ USE THE DERIVATIVE OF A FUNCTION TO FIND EQUATIONS OF TANGENT LINES
- ☐ INTERPRET GRAPHS
  - ☐ ESTIMATE DERIVATIVE VALUES
  - ☐ IDENTIFY POINTS WHERE f IS NOT DIFFERENTIABLE

NOTATION

- $f'(x)$: READ "F-PRIME OF x"
- $\frac{dy}{dx}$: DIFFERENTIAL NOTATION
- y'

d : REFERS TO INFINITELY SMALL CHANGE.

Δ : CHANGE

DEFINITION

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

* f' IS A FUNCTION WHOSE OUTPUT IS THE SLOPE OF THE TANGENT TO f.

- WE MUST EVALUATE f' AT AN x-VALUE TO GET A SLOPE.

EX: $f(x) = 2\sqrt{x}$. COMPUTE f' AND USE IT TO GET TANGENT LINE AT $x = 9$.

$$f'(x) = \lim_{h\to 0} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h} \cdot \frac{2\sqrt{x+h} + 2\sqrt{x}}{2\sqrt{x+h} + 2\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{4(x+h) - 4(x)}{h(2\sqrt{x+h} + 2\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{4h}{h(2\sqrt{x+h} + 2\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{4}{2\sqrt{x+h} + 2\sqrt{x}}$$

$$= \frac{4}{2\sqrt{x} + 2\sqrt{x}}$$

$$= \frac{4}{4\sqrt{x}}$$

$$= \boxed{\frac{1}{\sqrt{x}}}$$

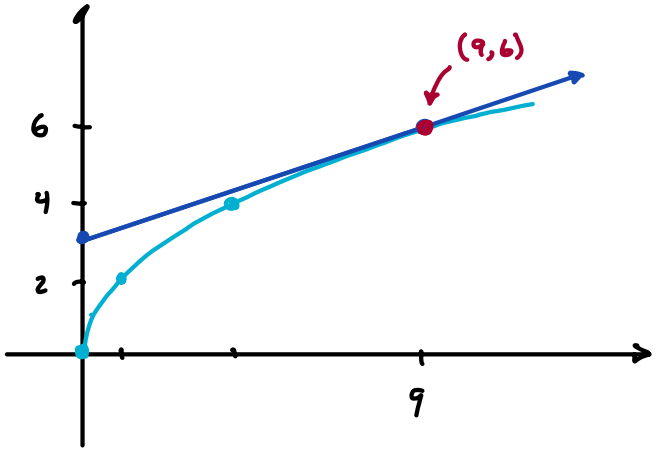
TANGENT: $x = 9$ (GIVEN)

$y = f(9) = 2\sqrt{9} = 6$

$m = f'(9) = \frac{1}{\sqrt{9}} = \frac{1}{3}$

$$y - 6 = \frac{1}{3}(x - 9)$$

$$\boxed{y = \frac{1}{3}x + 3}$$



DIFFERENTIABILITY IF f' IS CONTINUOUS, WE SAY f IS DIFFERENTIABLE.

e.g. $f(x) = |x| = \begin{cases} x \; ; & x \geq 0 \\ -x \; ; & x < 0 \end{cases}$

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$x > 0$:

$$f'(x) = \lim_{h \to 0} \frac{x+h-x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \boxed{1}$$

$x < 0$:

$$f'(x) = \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h}$$

$$= \boxed{-1}$$

$x = 0$

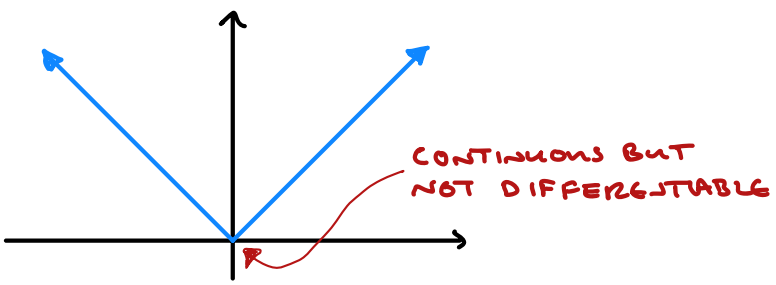
$$f'(x) = \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

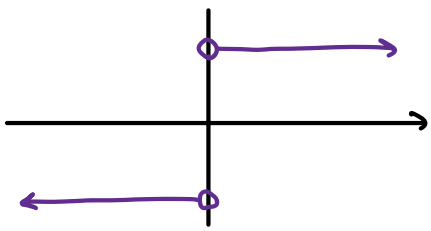
DNE.

$f'(0)$ IS UNDEFINED

$f(x) = |x|$

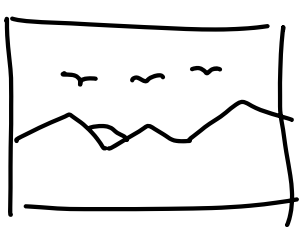


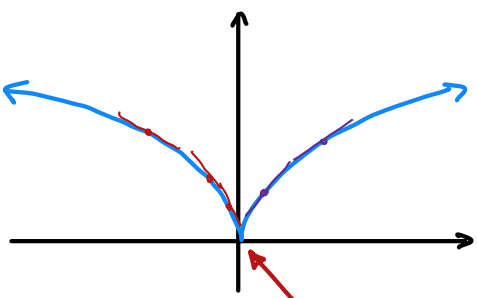
$$f'(x) = \begin{cases} 1 \; ; & x > 0 \\ -1 \; ; & x < 0 \end{cases}$$



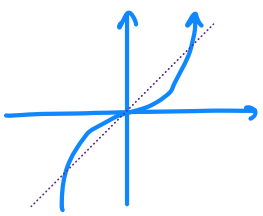
* WHEN f IS DIFFERENTIABLE ITS GRAPH IS SMOOTH (NO CORNERS OR CUSPS).

CUSP: $y = x^{2/3}$

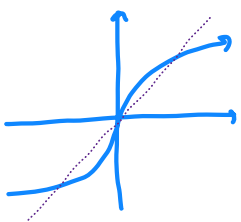




$y = x^3$



$y = x^{1/3}$



EX: USING THE LIMIT DEFINITION, COMPUTE THE DERIVATIVE OF $f(x) = 3x^2 - x$

$$f(x+h) = 3(x+h)^2 - (x+h)$$

$$= 3(x^2 + 2xh + h^2) - x - h$$

$$= 3x^2 + 6xh + 3h^2 - x - h$$

$$f(x) = 3x^2 - x$$

$$f'(x) = \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 - x - h) - (3x^2 - x)}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - h}{h}$$

$$= \lim_{h \to 0} (6x + 3h - 1)$$

$$= \boxed{6x - 1}$$

EX: USING THE LIMIT DEFINITION, COMPUTE THE DERIVATIVE OF $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \frac{-1}{x(x)}$$

$$= \boxed{-1/x^2}$$

Ex: PUT THE FOLLOWING IN ORDER: $f'(0), f'(1), f'(3), f'(5), f'(6)$

