MATH 121

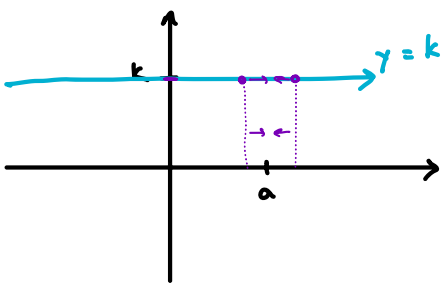
COMPUTING LIMITS

GOALS

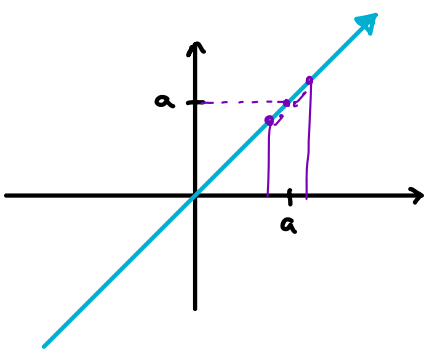
- ☐ GIVEN A FUNCTION, COMPUTE THE ONE-SIDED AND TWO-SIDED LIMITS AS x APPROACHES A FINITE VALUE
- ☐ KNOW WHAT TO DO FOR EACH OF THREE CASES FOR NON PIECEWISE FUNCTIONS
  1. PLUG IN AND DONE
  2. GET $\frac{0}{0}$
  3. GET $\frac{k}{0}$ FOR $k \neq 0$
- ☐ APPLY THE THREE CASES AND ONE ISTUITIVE APPROACH TO PIECEWISE FUNCTIONS

SOME IMPORTANT PROPERTIES OF LIMITS

1. $\lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = \lim_{x \to a} \left(f(x) \pm g(x) \right)$
2. $\lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = \lim_{x \to a} \left(f(x) g(x) \right)$
3. $\lim_{x \to a} f(x) \Big/ \lim_{x \to a} g(x) = \lim_{x \to a} \left(\frac{f(x)}{g(x)} \right)$ $\quad \left[\lim_{x \to a} g(x) \neq 0 \right]$
4. $\left(\lim_{x \to a} f(x) \right)^n = \lim_{x \to a} \left(f(x) \right)^n$
5. $\lim_{x \to a} k = k$



6. $\lim_{x \to a} x = a$



EX SUPPOSE $\lim_{x\to a} f(x) = 4$ AND $\lim_{x\to a} g(x) = 7$

1. $\lim_{x\to a}\left(f(x)+g(x)\right) = 4+7 = \boxed{11}$

2. $\lim_{x\to a} 2f(x) = 2\cdot 4 = \boxed{8}$

3. $\lim_{x\to a}\left(3f(x)-2g(x)\right) = 3\cdot 4 - 2\cdot 7 = \boxed{-2}$

THREE CASES $\lim_{x\to a} f(a)$ WHERE f IS NOT PIECEWISE

1. $f(a)$ IS DEFINED (I.E. WE CAN PLUG IN $x=a$ DIRECTLY)

2. PLUG IN $x=a$ AND GET $\frac{0}{0}$

3. PLUG IN $x=a$ AND GET $\frac{k}{0}$ $(k\neq 0)$

CASE 1

IF $f(a)$ IS DEFINED, THEN $\lim_{x\to a} f(x) = f(a)$

EXAMPLES

1. $\lim_{x\to 3}(x^2+1) = 3^2+1 = \boxed{10}$

2. $\lim_{x\to -1} \ln(x+2) = \ln(-1+2) = \ln(1) = \boxed{0}$

3. $\lim_{x\to \pi} \sin(x/2) = \sin \pi/2 = \boxed{1}$

4. $\lim_{x\to 0} e^x = e^0 = \boxed{1}$

5. $\lim_{x\to \pi/4} \sec x = \sec\frac{\pi}{4} = \boxed{\sqrt{2}}$

$\ln A = B$

$\Updownarrow$

$e^B = A$

$\ln 1 = ?$

$e^? = 1$

CASE 2

IF WE PLUG $x = a$ AND GET $\frac{0}{0}$, THEN WE CAN SIMPLIFY BY FACTORING, MULTIPLYING BY CONJUGATE, OR SOME OTHER TRICKS.

EXAMPLES

1. $\lim_{x \to 4} \frac{x - 4 \to 0}{x^2 - 16 \to 0} = \lim \frac{x - 4}{(x+4)(x-4)}$

$= \lim_{x \to 4} \frac{1}{x+4}$ CASE 1

$= \frac{1}{4+4}$

$= \boxed{\frac{1}{8}}$

DIFF. OF SQ'S

$A^2 - B^2 = (A+B)(A-B)$

2. $\lim_{x \to -1} \frac{x^2 + 3x + 2}{x^2 - 2x - 3} = \lim_{x \to -1} \frac{(x+2)(x+1)}{(x-3)(x+1)}$

$= \lim_{x \to -1} \frac{x+2}{x-3}$ CASE 1

$= \frac{-1+2}{-1-3}$

$= \boxed{-1/4}$

3. $\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 9} = \lim_{x \to 3} \frac{(x-3)(x+2)}{(x-3)(x+3)}$

$= \lim_{x \to 3} \frac{x+2}{x+3}$

$= \boxed{5/6}$

4. $\lim_{x \to 9} \dfrac{x-9}{\sqrt{x}-3}$ $\to \dfrac{0}{0}$

TWO OPTIONS FOR HOW TO HANDLE THIS

i. TREAT $x-9$ LIKE A DIFF. OF SQUARES

ii. MULT. BY THE CONJUGATE OF $\sqrt{x}-3$ ON TOP & BOTTOM.

↳ A+B vs. A-B

i. $x - 9 = (\sqrt{x})^2 - (3)^2$

$= (\sqrt{x}+3)(\sqrt{x}-3)$

$$\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \to 9} \frac{(\sqrt{x}+3)(\sqrt{x}-3)}{\sqrt{x}-3}$$

$$= \lim_{x \to 9} (\sqrt{x}+3)$$

$$= \sqrt{9} + 3$$

$$= \boxed{6}$$

ii.

$$\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x \to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9}$$

DON'T DISTRIBUTE UP TOP

$x\sqrt{x} + 3x - 9\sqrt{x} - 27$!!

$$= \lim_{x \to 9} \sqrt{x}+3$$

$$= \boxed{6}$$

5. $\lim_{x \to 4} \dfrac{\sqrt{x}-2}{x-4} = \lim_{x \to 4} \dfrac{\sqrt{x}-2}{(\sqrt{x}+2)(\sqrt{x}-2)}$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x}+2}$$

$$= \boxed{\frac{1}{4}}$$

CASE 3

IF WE PLUG $x = a$ AND GET $\frac{k}{0}$ $(k \neq 0)$, THEN THE GRAPH OF f HAS A **VERTICAL ASYMPTOTE AT $x = a$.**

* THIS MEANS $\lim_{x \to a^-} f(x) = \pm\infty$ AND $\lim_{x \to a^+} f(x) = \pm\infty$.

WE CAN USE SIGN CHARTS TO FIGURE OUT THE SIGNS.

EXAMPLES

1. $\lim_{x \to 2^-} \frac{3x-5}{x-2} \begin{matrix} \to 1 \\ \to 0 \end{matrix} = \boxed{-\infty}$

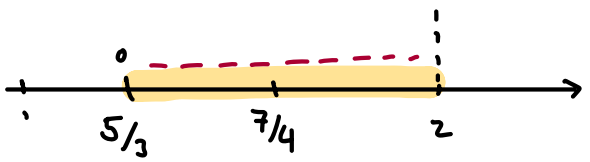
$\frac{1}{0}$ TELLS US THERE'S A VERT. ASY. AT $x = 2$. (SO LEFT AND RIGHT LIMITS ARE $\pm\infty$)

FIND POINTS WHERE TOP = 0; BOTTOM = 0

$3x - 5 = 0 \longrightarrow \boxed{x = 5/3}$

$x - 2 = 0 \longrightarrow \boxed{x = 2}$

ONLY PLACES WHERE f CAN CHANGE SIGN



$\frac{3(7/4) - 5}{7/4 - 2}$ $\begin{matrix} (+) \\ (-) \end{matrix}$ $\Big\}$ $(-)$

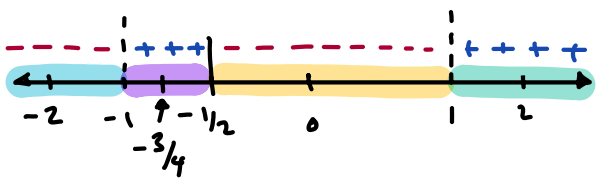
2. $\lim_{x \to -1^-} \frac{2x+1}{x^2-1} \begin{matrix} \to -1 \\ \to 0 \end{matrix} = -\infty$

$\lim_{x \to -1^+} \frac{2x+1}{x^2-1} \begin{matrix} \to -1 \\ \to 0 \end{matrix} = +\infty$

$\lim_{x \to 1^-} \frac{2x+1}{x^2-1} \begin{matrix} \to 3 \\ \to 0 \end{matrix} = -\infty$

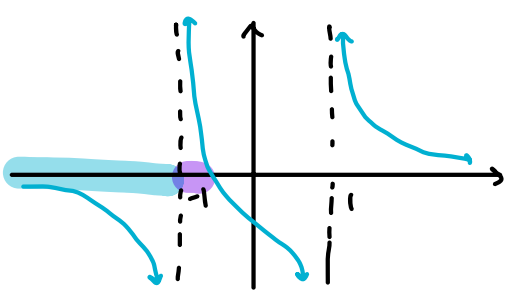
$\lim_{x \to 1^+} \frac{2x+1}{x^2-1} \begin{matrix} \to 3 \\ \to 0 \end{matrix} = +\infty$

TOP = 0	BOTTOM = 0
$2x + 1 = 0$	$x^2 - 1 = 0$
$x = -1/2$	$x = \pm 1$



$\frac{2(-2)+1}{(-2)^2-1}$ $\begin{matrix} (-) \\ (+) \end{matrix}$ $\qquad$ $\frac{2(-3/4)+1}{(-3/4)^2-1}$ $\begin{matrix} (-) \\ (-) \end{matrix}$

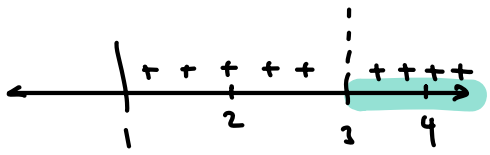
$\frac{2(0)+1}{0^2-1}$ $\begin{matrix} (+) \\ (-) \end{matrix}$ $\qquad$ $\frac{2(2)+1}{2^2-1}$ $\begin{matrix} (+) \\ (+) \end{matrix}$



* NOTE: SIGNS WON'T ALWAYS ALTERNATE

SOME MORE EXAMPLES

1. $\lim_{x \to 3^+} \dfrac{x-1}{|x-3|} = \boxed{+\infty}$



$$\frac{4-1}{|4-3|}$$

2. $\lim_{x \to 1} \dfrac{x^2+3x+4}{x^2-x-2} = \dfrac{1^2+3(1)+4}{1^2-1-2} = \dfrac{8}{-2} = \boxed{-4}$

3. $\lim_{x \to -2} \dfrac{x^2+5x+6}{x^2+6x+8} \begin{matrix}\to 0 \\ \to 0\end{matrix} = \lim_{x \to -2} \dfrac{(x+3)(x+2)}{(x+4)(x+2)}$

$$= \lim_{x \to -2} \frac{(x+3)}{(x+4)}$$

$$= \boxed{\frac{1}{2}}$$

4. $\lim_{x \to 25} \dfrac{\sqrt{x}-5}{x-25} \begin{matrix}\to 0 \\ \to 0\end{matrix} = \lim_{x \to 25} \dfrac{\sqrt{x}-5}{(\sqrt{x}+5)(\sqrt{x}-5)}$

$$= \lim_{x \to 25} \frac{1}{\sqrt{x}+5}$$

$$= \boxed{\frac{1}{10}}$$

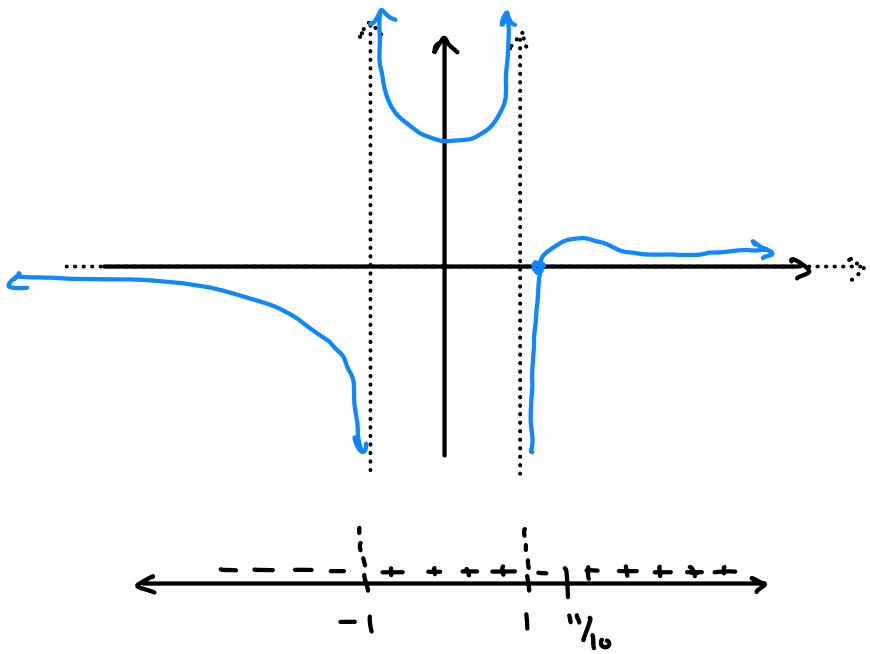
5.

$$\lim_{x \to 1^+} \frac{10x - 11}{x^2 - 1} = -\infty$$

$$\lim_{x \to 1^-} \frac{10x - 11}{x^2 - 1} = +\infty$$

$$\lim_{x \to -1^-} \frac{10x - 11}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{10x - 11}{x^2 - 1} = +\infty$$

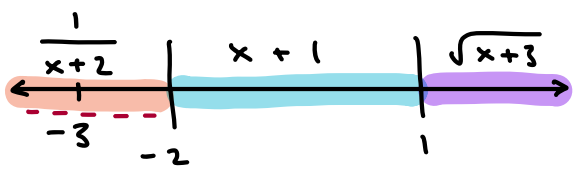


WORKING W/ PIECEWISE FUNCTIONS

$$f(x) = \begin{cases} g(x) ; & x \le a \\ h(x) ; & x > a. \end{cases}$$

e.g.

$$f(x) = \begin{cases} \frac{1}{x+2} ; & x < -2 \\ x + 1 ; & -2 \le x < 1 \\ \sqrt{x+3} ; & x \ge 1 \end{cases}$$



$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{1}{x+2} \quad (\to 1 / \to 0)$$

CASE 3

$$= \boxed{-\infty}$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} x + 1$$

$$= \boxed{-1}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x + 1$$

$$= \boxed{2}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x+3}$$

$$= \boxed{2}$$

$$\boxed{\lim_{x \to 1} f(x) = 2}$$