MATH 121
CHAIN RULE

GOALS

- ☐ DIFFERENTIATE COMPOSITIONS OF FUNCTIONS USING THE CHAIN RULE
  - ☐ EXTEND TO COMPOSITIONS OF THREE OR MORE FUNCTIONS
- ☐ USE THE CHAIN RULE IN CONJUNCTION WITH OTHER RULES FOR COMBINING DERIVATIVES: CONSTANT MULTIPLE, SUM/DIFFERENCE, PRODUCT, AND QUOTIENT

WARM UP

COMPUTE THE DERIVATIVE OF EACH FUNCTION

1. $f(x) = 6\sqrt[3]{x^2}$

REWRITE f: $f(x) = 6x^{2/3}$

POWER RULE: $f'(x) = 6\left(\frac{2}{3}x^{-1/3}\right)$

$$= \boxed{\frac{4}{\sqrt[3]{x}}}$$

2. $g(x) = 3x^4 \sec(x)$

PRODUCT RULE: $a = 3x^4$ $b = \sec(x)$

$$\boxed{g'(x) = \underset{a'}{12x^3}\underset{b}{\sec(x)} + \underset{a}{3x^4}\underset{b'}{\sec(x)\tan(x)}}$$

3. $h(x) = \dfrac{4x^2 + x - 3}{x^2}$

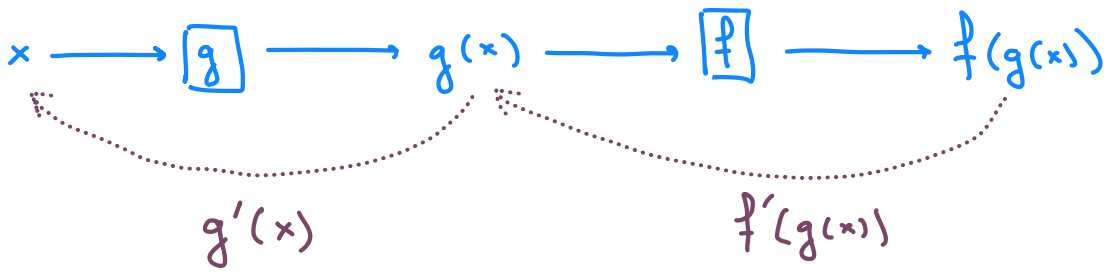
REWRITE h: $h(x) = 4 + x^{-1} - 3x^{-2}$

$$h'(x) = -x^{-2} + 6x^{-3}$$

$$= \boxed{-\frac{1}{x^2} + \frac{6}{x^3}}$$

CHAIN RULE

ALLOWS US TO DIFFERENTIATE **COMPOSITIONS OF FUNCTIONS**.



$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

* WE CAN'T DIFFERENTIATE OUTSIDE AND INSIDE AT THE SAME TIME *

IT'S NOT ~~$f'(g'(x))$~~ ∴

IF IT WERE... ~~$(\sin(x))' = \cos(1)$~~ ¡NONSENSE!

EX: $y = \sin(3x^4) = f(g(x))$

$f(x) = \sin x \longrightarrow f'(x) = \cos x$

$y' = f'(g(x)) \cdot g'(x)$

$g(x) = 3x^4 \longrightarrow g'(x) = 12x^3$

$y' = \cos(3x^4) \cdot 12x^3$

$= \boxed{12x^3 \cos(3x^4)}$

EX: $y = 2(x^3+1)^5 = f(g(x))$

$f(x) = 2x^5 \longrightarrow f'(x) = 10x^4$

$y' = f'(g(x)) \cdot g'(x)$

$g(x) = x^3 + 1 \longrightarrow g'(x) = 3x^2$

$y' = 10(x^3+1)^4 \cdot 3x^2$

$= \boxed{30x^2(x^3+1)^4}$

Ex: $y = \sec(5x^4)$ $\quad A = 5x^4$

$y = \sec A \longrightarrow y' = \sec A \tan A \cdot A'$ (SHORTHAND)

$y' = \sec(5x^4)\tan(5x^4) \cdot 20x^3$ $\quad A'$

Ex: $y = \tan^4(x) = (\tan(x))^4$ $\quad A = \tan x$

$y = A^4 \longrightarrow y' = 4A^3 \cdot A'$

$y' = 4(\tan x)^3 \cdot \sec^2 x$

$= \boxed{4\tan^3 x \sec^2 x}$

Ex: $y = \tan^4(x^2+1) = [\tan(x^2+1)]^4$

$y = 4[\tan(x^2+1)]^3 \sec^2(x^2+1) \cdot 2x$

$= \boxed{8x\tan^3(x^2+1)\sec^2(x^2+1)}$

$f(x) = x^4 \quad g(x) = \tan x \quad h(x) = x^2+1$

$y = f(g(h(x)))$

$y' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$

LAYERS

✓ $[\ ]^4 \longrightarrow 4[\ ]^3$

✓ $\tan(\ ) \longrightarrow \sec^2(\ )$

✓ $x^2+1 \longrightarrow 2x$

TRY $\quad \cos(A)$

① $y = \cos(5x^4 + x)$

$y = -\sin(5x^4+x)(20x^3+1)$

$-\sin(A) \cdot A'$

② $y = \sqrt{6x^2+1} = (6x^2+1)^{1/2}$

$y' = \frac{1}{2}(6x^2+1)^{-1/2} \, 12x = \frac{6x}{\sqrt{6x^2+1}}$

③ $y = \sec^2(3x+1)$

$= \left(\sec(3x+1)\right)^2$

$y' = 2\left(\sec(3x+1)\right)\sec(3x+1)\tan(3x+1)\cdot 3$

$= \boxed{6\sec^2(3x+1)\tan(3x+1)}$

TRY

① $y = \overbrace{x^4}^{A}\overbrace{\sin(x^2+1)}^{B}$

$y' = \underset{A'}{4x^3}\underset{B}{\sin(x^2+1)} + \underset{A}{x^4}\underbrace{\cos(x^2+1)\cdot 2x}_{B'}$ (CHAIN RULE!)

$= \underset{A'}{4x^3}\underset{B}{\sin(x^2+1)} + \underset{A}{2x^5}\underset{B'}{\cos(x^2+1)}$

② $y = \dfrac{5}{(x^3+4)^4} = 5(x^3+4)^{-4}$

ANY QUOTIENT CAN BE TREATED AS A PRODUCT, BUT IT'S REALLY USEFUL WHEN THE NUMERATOR IS CONSTANT

$y' = -20(x^3+4)^{-5}\cdot 3x^2$

$= \dfrac{-60x^2}{(x^3+4)^5}$

QUOTIENT:

$y' = \dfrac{(x^3+4)^4\cdot(0) - 4(x^3+4)^3\cdot 3x^2\cdot 5}{(x^3+4)^8}$

③ $y = \csc^3(5x-1) = \left[\csc(5x-1)\right]^3$

$y' = 3\left[\csc(5x-1)\right]^2\left(-\csc(5x-1)\cot(5x-1)\right)(5)$

$= -15\csc^3(5x-1)\cot(5x-1)$