MATH 122

AREA BETWEEN CURVES

GOALS

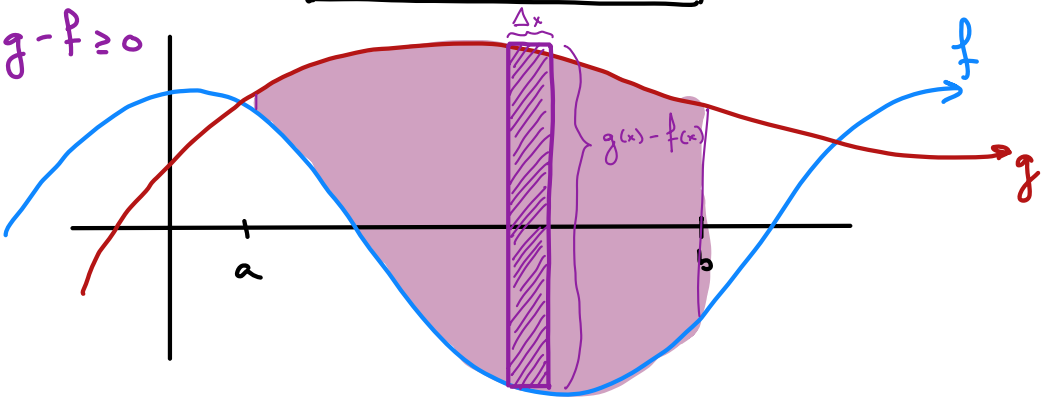
- ☐ SKETCH, SET UP, AND COMPUTE THE AREA BETWEEN TWO CURVES
- ☐ INTEGRATE IN TERMS OF y TO COMPUTE AREA.

MAIN IDEA: IF f AND g ARE CONTINUOUS WITH $g(x) \geq f(x)$ ON $[a,b]$, THEN THE AREA ENCLOSED BY $y = f(x)$, $y = g(x)$, $x = a$, AND $x = b$ IS

$$\int_a^b \underbrace{g(x) - f(x)}_{\geq 0}\, dx$$

NOT NET-SIGNED AREA BUT TRUE AREA:

$g \geq f \rightarrow g - f \geq 0$



EX: FIND THE AREA ENCLOSED BY $y = x^2$ AND $y = 2x$

WITHOUT AN INTERVAL, IT'S OUR JOB TO FIGURE OUT $[a,b]$.

INTERSECTIONS:

$$x^2 = 2x$$

NEVER DIVIDE IN SITUATIONS LIKE THIS!

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

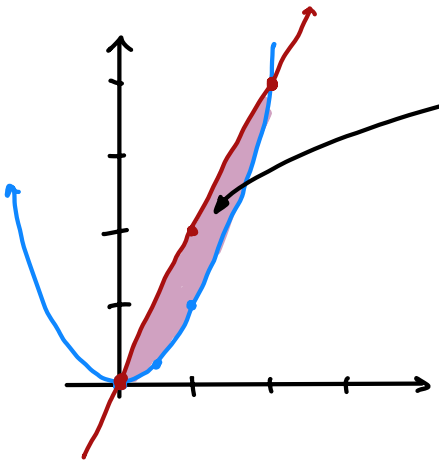
$$\boxed{x = 0, 2}$$

WHICH IS ON TOP IN $[0,2]$?

PLUG IN $x = 1$ (ANYTHING IN $[0,2]$)

$(1)^2 = 1$ VS. $2(1) = 2$

BOTTOM: $y = x^2$ TOP: $y = 2x$



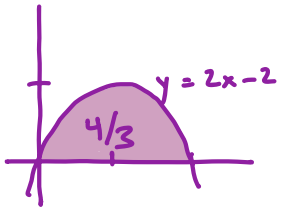
AREA: $\int_0^2 2x - x^2 \, dx$

$= x^2 - \frac{1}{3}x^3 \Big|_0^2$

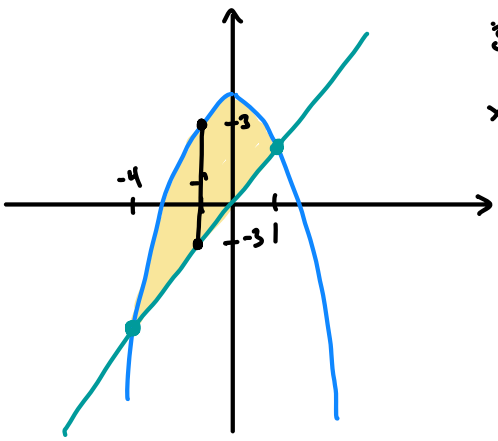
$= \left(4 - \frac{8}{3}\right) - (0 - 0)$

$= \boxed{4/3}$

NOTE: THE SAME INTEGRAL ALSO GIVES THE NET-SIGNED AREA FOR $y = 2x - x^2$



EX: SET UP AN INTEGRAL FOR THE AREA ENCLOSED BY $y = 4 - x^2$ AND $y = 3x$.



$3x = 4 - x^2$

$x^2 + 3x - 4 = 0$

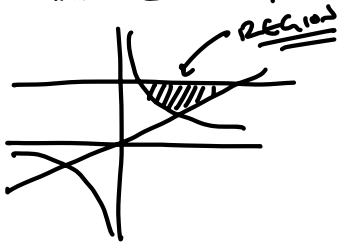
$(x+4)(x-1) = 0$

$\boxed{x = -4, 1}$

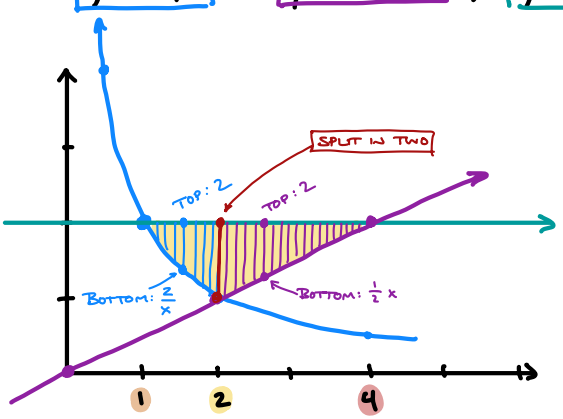
$$\boxed{A = \int_{-4}^{1} (4 - x^2) - (3x) \, dx}$$

EX: SET UP AN INTEGRAL FOR AREA ENCLOSED BY...

INITIAL SKETCH



$\boxed{y = \frac{2}{x}}$; $\boxed{y = \frac{1}{2}x}$; $\boxed{y = 2}$

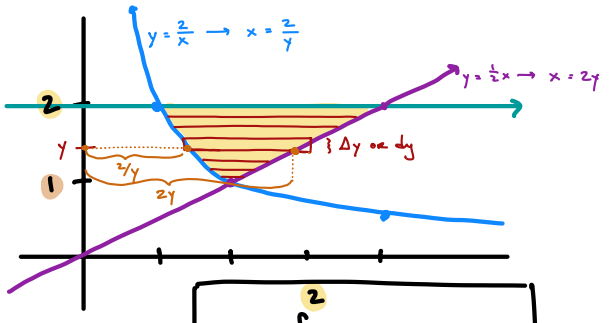


PAIRWISE INTERSECTIONS

- $\frac{2}{x} = \frac{1}{2}x \longrightarrow x^2 = 4 \longrightarrow x = 2$
- $\frac{2}{x} = 2 \longrightarrow x = 1$
- $\frac{1}{2}x = 2 \longrightarrow x = 4$

$$\boxed{A = \int_1^2 \underbrace{2}_{\text{TOP}} - \underbrace{\frac{2}{x}}_{\text{BOTTOM}} \, dx + \int_2^4 \underbrace{2}_{\text{TOP}} - \underbrace{\frac{1}{2}x}_{\text{BOTTOM}} \, dx}$$

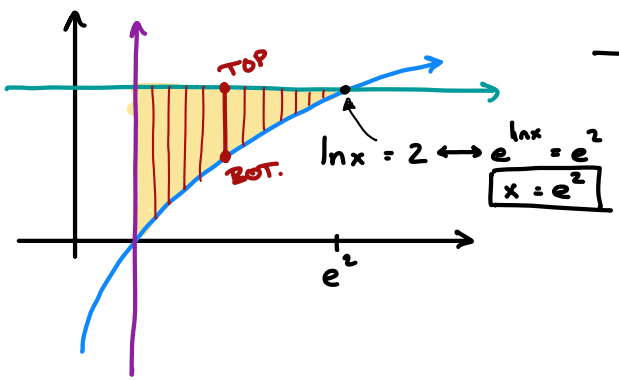
BOUNDS ARE x-VALUES



$$A = \int_1^2 2y - \frac{2}{y}\, dy$$

BOUNDS ARE y-VALUES

Ex: REGION ENCLOSED BY $y = \ln x$; $y = 2$; $x = 1$



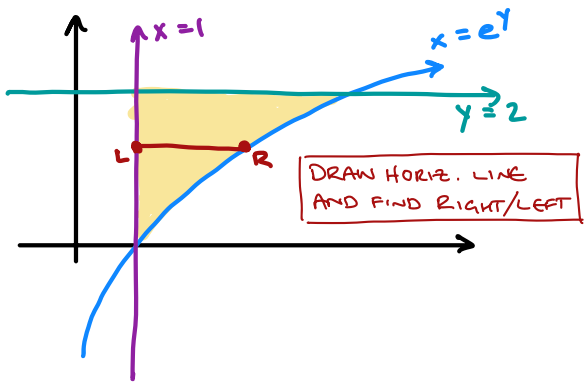
dx SETUP:

TOP: $y = 2$

BOTTOM: $y = \ln x$

$$A = \int_1^{e^2} 2 - \ln x\, dx$$

$\int \ln x\, dx = ??$



dy SETUP:

$y = \ln x \rightarrow x = e^y$

$$\int_0^2 e^y - 1\, dy$$

$$= e^y - y \Big|_0^2$$

$$= (e^2 - 2) - (1 - 0)$$

$$= \boxed{e^2 - 3}$$

EX: REGION ENCLOSED BY $y = \sqrt{x}$ AND $y = \frac{1}{2}x$.

$\sqrt{x} = \frac{1}{2}x$

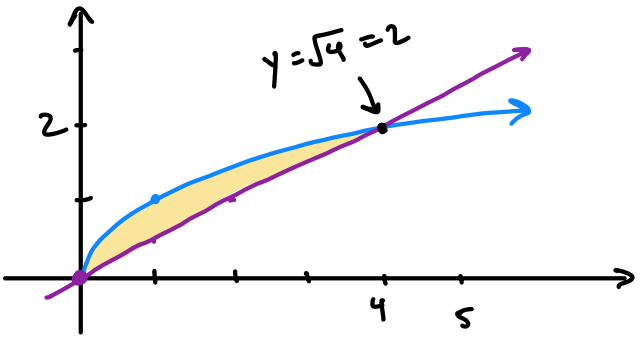
$x = \frac{1}{4}x^2$

$\frac{1}{4}x^2 - x = 0$

$\frac{1}{4}x(x-4) = 0$

$x = 0, 4$

- SET UP AS dx INTEGRAL
- SET UP AS dy INTEGRAL



$$\int_0^4 \sqrt{x} - \frac{1}{2}x \, dx$$

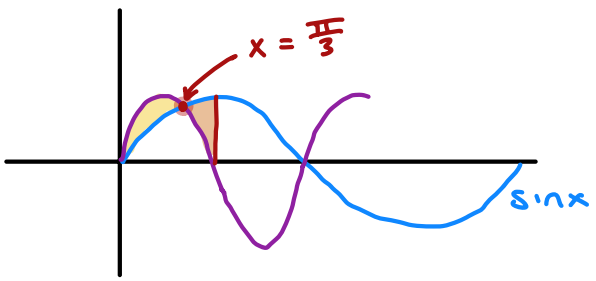
$y = \sqrt{x} \rightarrow x = y^2$; $y = \frac{1}{2}x \rightarrow x = 2y$

$y^2 = 2y$

$y^2 - 2y = 0 \rightarrow y = 0, 2$

$$\int_0^2 2y - y^2 \, dy$$

EX: AREA BETWEEN $y = \sin x$ AND $y = \sin 2x$ OVER $[0, \pi/2]$



$$\int_0^{\pi/3} \sin 2x - \sin x \, dx + \int_{\pi/3}^{\pi/2} \sin x - \sin 2x \, dx$$