MATH 122

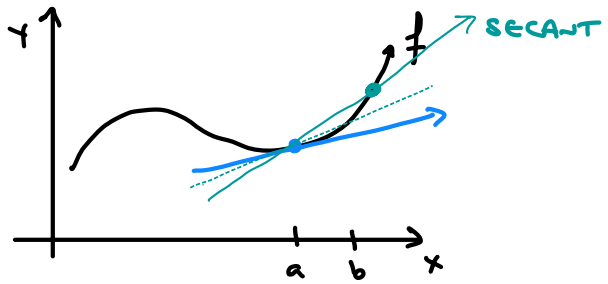
AREA AS A LIMIT

GOALS

- ☐ APPROXIMATE THE AREA UNDER A CURVE USING A GIVEN NUMBER OF RECTANGLES.
  - ☐ KNOW HOW TO DO SO WITH LEFT ENDPOINTS, RIGHT ENDPOINTS AND MID POINTS.
- ☐ WORK WITH SUMMATION NOTATION
  - ☐ EXPAND A GIVEN SUM
  - ☐ WRITE A SUM IN SIGMA NOTATION
  - ☐ APPLY SUMMATION FORMULAS TO COMPUTE LARGE SUMS
- ☐ USE RIEMANN SUMS TO COMPUTE EXACT AREA UNDER A CURVE
  - ☐ KNOW THE FORMULA AND INDIVIDUAL TERMS $(\Delta x, x_k^*)$

CALC 1 RECAP:

Q: HOW DO WE COMPUTE THE SLOPE OF THE TANGENT LINE TO f AT $x=a$?



$$m = \frac{f(b) - f(a)}{b - a}$$

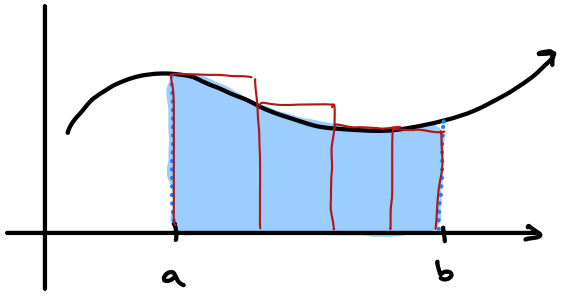
AS $b \to a$, SECANT $\longrightarrow$ TANGENT

$$m = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} = f'(a)$$

- START W/ AN APPROXIMATION
- USE A LIMIT TO MAKE IT EXACT.

CALC 2:

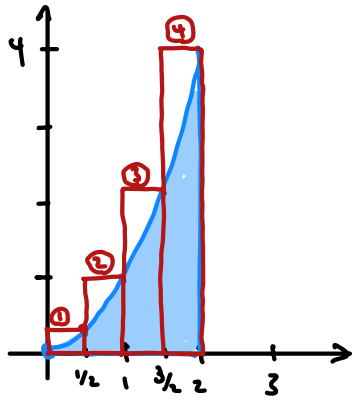
Q: GIVEN A CONTINUOUS f, HOW DO WE COMPUTE THE AREA BETWEEN THE GRAPH OF f AND THE x-AXIS OVER $[a,b]$?



- APPROXIMATION: FILL AREA w/ RECTANGLES (EASIEST AREA)

  EXACT: LET NUMBER OF RECT. GO TO INFINITY (LIMITS)

EX: $f(x) = x^2$; $[0,2]$; USE 4 RECTANGLES



RECTANGLES:

- EQUAL WIDTHS: EACH WIDTH (CALLED Δx) IS $\frac{1}{2}$ B/C $\frac{2-0}{4} = \frac{1}{2}$ (LENGTH OF INTERVAL $[0,2]$ / NUM. OF RECT.)

- USE TOP-RIGHT CORNER TO DETERMINE HEIGHTS.

$$A \approx ① + ② + ③ + ④$$

$$\approx \frac{1}{2} f(1/2) + \frac{1}{2} f(1) + \frac{1}{2} f(3/2) + \frac{1}{2} f(2) \quad \left| \text{USE } f(x) = x^2 \right.$$

$$\approx \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{2}(1) + \frac{1}{2}\left(\frac{9}{4}\right) + \frac{1}{2}(4)$$

$$\approx \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2$$

$$\approx \frac{30}{8}$$

$$\approx \boxed{\frac{15}{4}}$$

SUMMATION / SIGMA NOTATION

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + f(3) + \cdots + f(n)$$

CALLED "OPEN FORM"

k: CALLED THE "INDEX" (VARIABLE INSIDE SUM)

* k MUST BE AN INTEGER *

START k AT 1

n: LAST VALUE OF k.

e.g. $\sum_{k=1}^{5} (2k-1) = (2(1)-1) + (2(2)-1) + (2(3)-1) + (2(4)-1) + (2(5)-1)$

$f(k)$

$f(1) + f(2) + f(3) + f(4) + f(5)$

$= 1 + 3 + 5 + 7 + 9 = \boxed{25}$

EX: WRITE $1 + 4 + 9 + 16 + 25 + 36 + 49$ IN Σ-NOTATION

$$\boxed{\sum_{k=1}^{7} k^2}$$

EX: WRITE $6 - 9 + 12 - 15 + 18 - 21 + 24 - 27$ IN Σ-NOTATION

$$\boxed{\sum_{k=2}^{9} (-1)^k 3k = \sum_{k=2}^{9} \cos(\pi k) 3k}$$

$\sum_{k=2}^{9} 3k = 6 + 9 + \cdots + 27$

ALL POSITIVE.

Q: HOW TO ALTERNATE SIGNS?

EX: WRITE $1 + 3 + 5 + 7 + \cdots + 21$ IN Σ-NOTATION

$$\boxed{\sum_{k=1}^{10} (2k+1)}$$

PROPERTIES:

① $\sum_{k=1}^{n} (f(k) + g(k)) = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k)$

$(f(1)+g(1)) + (f(2)+g(2)) + \cdots = f(1) + f(2) + \cdots + g(1) + g(2)$

ORDER DOESN'T MATTER!

② $\sum_{k=1}^{n} a f(k) = a \sum_{k=1}^{n} f(k)$

$a f(1) + a f(2) + \cdots + a f(n) = a(f(1) + \cdots + f(n))$

FORMULAS:

③ $\sum_{k=1}^{n} 1 = n$ ← $1+1+1+\cdots+1$ (n TIMES.)

④ $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$

$+ \sum_{k=1}^{n} k = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$

$2\sum_{k=1}^{n} k = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$

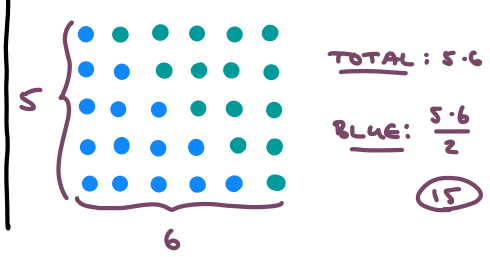
$2\sum k = n(n+1)$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

e.g. $\sum_{k=1}^{100} k = \frac{100 \cdot 101}{2} = 5050$

VISUAL "PROOF":

$\sum_{k=1}^{5} k$



⑤ $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

⑥ $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

$(2(1)-1) + (2(2)-1) + \cdots (2(24)-1)$
$1 + 3 + \cdots + 47 = 576$

Ex:

$$\sum_{k=1}^{24}(2k-1) = \sum_{k=1}^{24} 2k - \sum_{k=1}^{24} 1 \quad (1)$$

$$= 2\sum_{k=1}^{24} k - \sum_{k=1}^{24} 1 \quad (2)$$

(4) (3)

$$= 2 \cdot \frac{24 \cdot 25}{2} - 24$$

$$= 24 \cdot 25 - 24$$

$$= \boxed{576}$$

Ex: $\sum_{k=1}^{12}(k^2 - 2k) = \sum_{k=1}^{12} k^2 - 2\sum_{k=1}^{12} k$

$$= \frac{12(13)(25)}{6} - 2 \cdot \frac{12(13)}{2}$$

$$= \boxed{494}$$

Ex: $\sum_{k=4}^{20}(3k+1) = \sum_{k=1}^{20}(3k+1) - \sum_{k=1}^{3}(3k+1)$

$$= 3\sum_{k=1}^{20} k + \sum_{k=1}^{20} 1 - (4 + 7 + 10)$$

$$= 3\,\frac{20(21)}{2} + 20 - 21$$

$$= 630 - 1$$

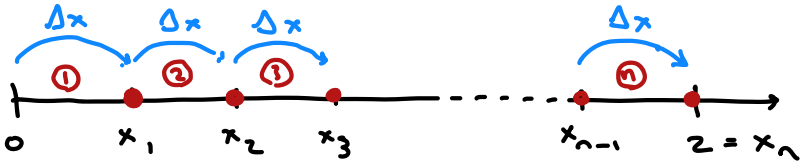
$$= \boxed{629}$$

EX: $f(x) = x^2$; $[0,2]$; USE n RECTANGLES W/ RIGHT ENDPOINTS.

$$\Delta x = \frac{2-0}{n}$$ (LENGTH OF INTERVAL) / (NUMBER OF RECT.)

$$\boxed{\Delta x = \frac{2}{n}}$$

X-VALUES: x_k^* OR x_k



COMPARE W/ WHAT WE DID WHEN $n=4$

$\Delta x = \frac{2}{4} = \frac{1}{2}$

$x_k = \frac{2k}{4} = \frac{k}{2}$

$x_1 = \frac{1}{2}$, $x_2 = 1$

$x_3 = \frac{3}{2}$; $x_4 = 2$

$$x_1 = 0 + \Delta x = \Delta x = \frac{2}{n}$$

$$x_2 = 0 + 2\Delta x = 2\Delta x = \frac{4}{n}$$

$$\vdots$$

$$x_k = 0 + k\Delta x = k\Delta x = \frac{2k}{n}$$

$$\vdots$$

$$x_n = \frac{2n}{n} = \boxed{2} \checkmark$$

HEIGHTS: PLUG x_k INTO GIVEN FUNCTION $f(x)$

$$f(x_k) = (x_k)^2 = \left(\frac{2k}{n}\right)^2 = \boxed{\frac{4k^2}{n^2}}$$

ADD UP ALL n AREAS:

$$A \approx \sum_{k=1}^{n} \underbrace{\left(\frac{4k^2}{n^2}\right)}_{\text{HEIGHT}} \overbrace{\left(\frac{2}{n}\right)}^{\text{WIDTH}}$$

$$\approx \sum_{k=1}^{n} \frac{8k^2}{n^3}$$

* WE CAN FACTOR OUT ANY NON-k TERM *

$$\approx \frac{8}{n^3} \sum_{k=1}^{n} k^2$$ ← APPLY SUMMATION FORMULA FOR $\sum k^2$

$$\approx \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$A \approx 8 \cdot \frac{2n^3 + 3n^2 + n}{6n^3}$$

GIVES APPROX. AREA FOR ANY n-VALUE (FOR ANY NUMBER OF RECT)

TAKE LIMIT AS $n \to \infty$ TO GET EXACT AREA:

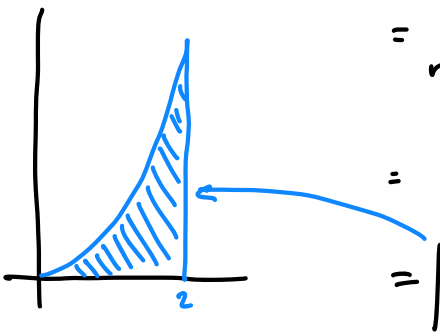
$$A = \lim_{n\to\infty} 8 \frac{2n^3 + 3n^2 + n}{6n^3}$$

TAKE LEADING TERMS

$$= \lim_{n\to\infty} 8 \cdot \frac{2n^3}{6n^3}$$

$$= 8 \cdot \frac{2}{6}$$

$$= \boxed{\frac{8}{3}}$$



IN GENERAL: THE NET-SIGNED AREA FOR f OVER $[a,b]$ IS GIVEN BY THE RIEMANN SUM

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

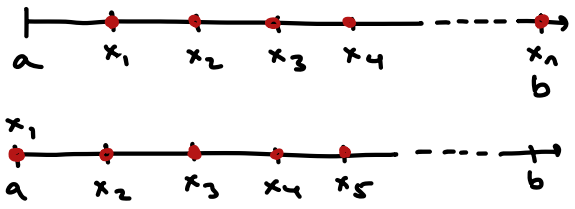
$\lim_{n\to\infty}$: LETS THE NUMBER OF RECTANGLES GO TO ∞

$\sum_{k=1}^{n}$: ADDS UP AREAS OF n RECTANGLES

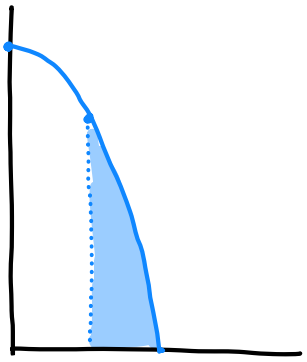
$f(x_k)$: HEIGHTS; Δx: WIDTHS

$$\boxed{\Delta x = \frac{b-a}{n}}$$

$$\boxed{x_K = \begin{cases} \text{RIGHT}: a + k\Delta x \\ \text{LEFT}: a + (k-1)\Delta x \\ \text{MID}: a + (k - \frac{1}{2})\Delta x \end{cases}}$$



Ex: $f(x) = 4 - x^2$; $[1, 2]$; RIGHT ENDPOINTS



$$\Delta x = \frac{2-1}{n} = \frac{1}{n}$$

$$x_k = a + k\Delta x$$
$$= 1 + k \cdot \frac{1}{n}$$
$$= 1 + \frac{k}{n}$$

$(1 + \frac{k}{n})(1 + \frac{k}{n})$

$$f(x_k) = 4 - \left(1 + \frac{k}{n}\right)^2$$
$$= 4 - \left(1 + \frac{2k}{n} + \frac{k^2}{n^2}\right)$$
$$= 3 - \frac{2k}{n} - \frac{k^2}{n^2}$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left(3 - \frac{2k}{n} - \frac{k^2}{n^2}\right)\left(\frac{1}{n}\right)$$

$$= \lim \sum \left(\frac{3}{n} - \frac{2k}{n^2} - \frac{k^2}{n^3}\right)$$

$$= \lim \left(\sum \frac{3}{n} - \sum \frac{2k}{n^2} - \sum \frac{k^2}{n^3}\right)$$

$$= \lim \left(\frac{3}{n}\sum 1 - \frac{2}{n^2}\sum k - \frac{1}{n^3}\sum k^2\right)$$

$$= \lim \left(\frac{3}{n}(n) - \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim \left(3 - \frac{n^2+n}{n^2} - \frac{2n^3+3n^2+n}{6n^3}\right)$$

$$= 3 - 1 - \frac{1}{3}$$

$$= \boxed{5/3}$$

* POWERS OF n IN DENOMINATORS ARE AT LEAST ONE GREATER THAN k POWERS IN NUMERATORS

ex: $f(x) = x^3$; $[0,1]$; LEFT ENDPOINTS

$$\Delta x = \frac{1}{n} \qquad x_k^* = 0 + (k-1)\frac{1}{n}$$

$$= \frac{k-1}{n}$$

$$f(x_k^*) = \left(\frac{k-1}{n}\right)^3 = \frac{k^3 - 3k^2 + 3k - 1}{n^3}$$

$$f(x_k^*) \cdot \Delta x = \frac{k^3 - 3k^2 + 3k - 1}{n^3} \cdot \frac{1}{n}$$

$$= \frac{k^3 - 3k^2 + 3k - 1}{n^4}$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3 - 3k^2 + 3k - 1}{n^4}$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3 - \frac{3}{n^4}\sum_{k=1}^{n} k^2 + \frac{3}{n^4}\sum_{k=1}^{n} k - \frac{1}{n^4}\sum_{k=1}^{n} 1$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{3}{n^4}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{3}{n^4}\left(\frac{n(n+1)}{2}\right) - \frac{1}{n^4}\cdot n$$

$$= \lim_{n\to\infty} \frac{n^4 + 2n^3 + n^2}{4n^4} - \frac{2n^3 + 3n^2 + n}{6n^4} + \frac{3(n^2+n)}{2n^4} - \frac{1}{n^3}$$

$$= \frac{1}{4} - 0 + 0 - 0$$

$$= \boxed{\frac{1}{4}}$$