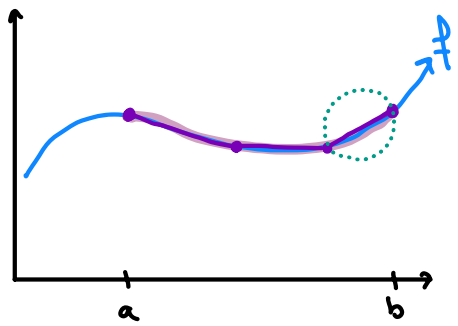
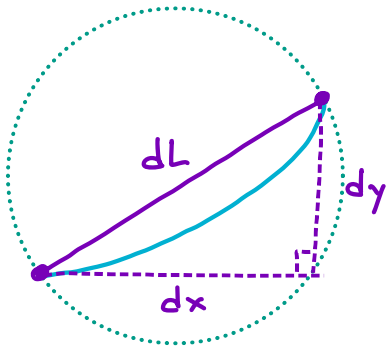
MATH 122

LENGTH OF A PLANE CURVE (ARC LENGTH)

GOALS

- ☐ KNOW AND UNDERSTAND THE FORMULA FOR THE LENGTH OF A PLANE CURVE
- ☐ KNOW THE COMMON "TRICKS" THAT COME UP W/ ARC LENGTH PROBLEMS





$$dx^2 + dy^2 = dL^2$$

$$dL = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$\int dL = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\boxed{L = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx}$$

DIST. FORMULA!

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$c = \sqrt{a^2 + b^2}$$

PYTH. THM.

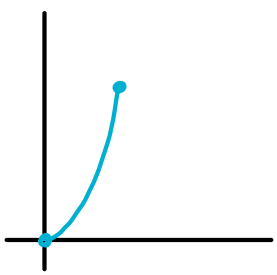
$$\frac{dy}{dx} = f'$$

Ex: $y = x^2$; $[0, 2]$

$$\frac{dy}{dx} = 2x$$

$$L = \int_0^2 \sqrt{1 + (2x)^2}\, dx$$

$$= \int_0^2 \sqrt{1 + 4x^2}\, dx$$



DON'T KNOW HOW TO DO THIS (YET)!!

Ex: $y = \frac{2}{3}(x^2-1)^{3/2}$; $[1,3]$

$$\frac{dy}{dx} = (x^2-1)^{1/2} \cdot 2x$$

$$\left(\frac{dy}{dx}\right)^2 = (x^2-1)\cdot 4x^2$$

$$= 4x^4 - 4x^2$$

$$L = \int_1^3 \sqrt{1 + 4x^4 - 4x^2}\, dx$$

$$= \int_1^3 \sqrt{4x^4 - 4x^2 + 1}\, dx$$

$$= \int_1^3 \sqrt{(2x^2-1)^2}\, dx$$

$$= \int_1^3 2x^2 - 1\, dx$$

$$= \frac{2}{3}x^3 - x \Big|_1^3 = (18-3) - \left(\frac{2}{3} - 1\right)$$

$$= \boxed{46/3}$$

Ex: $y = \frac{x^2}{2} - \frac{\ln x}{4}$; $[2,4]$

$$\frac{dy}{dx} = x - \frac{1}{4x} \quad \text{OR} \quad \frac{4x^2-1}{4x}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(x - \frac{1}{4x}\right)\left(x - \frac{1}{4x}\right)$$

$$= x^2 - \frac{1}{4} - \frac{1}{4} + \frac{1}{16x^2}$$

$$= x^2 - \frac{1}{2} + \frac{1}{16x^2}$$

OR

$$\frac{16x^4 - 8x^2 + 1}{16x^2}$$

$$L = \int_2^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_2^4 \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}}\, dx$$

LIKE TERMS

$$= \int_2^4 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}\, dx$$

$$= \int_2^4 \sqrt{\left(x + \frac{1}{4x}\right)^2}\, dx$$

$$= \int_2^4 x + \frac{1}{4x}\, dx$$

$$= \frac{x^2}{2} + \frac{1}{4}\ln x \Big|_2^4$$

$$= \left(8 + \frac{1}{4}\ln 4\right) - \left(2 + \frac{1}{4}\ln 2\right)$$

$$= 6 + \frac{1}{4}\ln 4 - \frac{1}{4}\ln 2$$

$$= \boxed{6 + \frac{1}{4}\ln 2}$$

Ex: $F(x) = \int_1^x \sqrt{t^4 - 1}\, dt$; $[1,2]$

$$F'(x) = \sqrt{x^4 - 1}$$

$$(F'(x))^2 = x^4 - 1$$

$$L = \int_1^2 \sqrt{1 + x^4 - 1}\, dx$$

$$= \int_1^2 \sqrt{x^4}\, dx$$

$$= \int_1^2 x^2\, dx$$

$$= \frac{x^3}{3}\Big|_1^2$$

$$= \boxed{7/3}$$

Ex: $x^2 = 9y^3$; FROM $(0,0)$ TO $(81,9)$

$$\boxed{x = 3y^{3/2}} \qquad y = \sqrt[3]{\frac{1}{9}}\, x^{2/3}$$

$$\frac{dx}{dy} = \frac{9}{2} y^{1/2}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{81}{4} y$$

$$L = \int_0^9 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^9 \sqrt{1 + \frac{81}{4} y}\, dy$$

$u = 1 + \frac{81}{4} y \rightarrow du = \frac{81}{4} dy$

$y = 0 \rightarrow u = 1$

$y = 9 \rightarrow u = \frac{283}{4}$

$$= \int_1^{\frac{283}{4}} u^{1/2}\, du.$$

$$= \frac{2}{3} u^{3/2}\Big|_1^{283/4}$$

$$= \text{☺}$$

Ex: $y = 4x^{3/2} - 1$ $\quad \left[\frac{1}{12}, \frac{2}{9}\right]$

$$\frac{dy}{dx} = 6x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = 36x$$

$$L = \int_{1/12}^{2/9} \sqrt{1 + 36x}\, dx$$

$u = 1 + 36x \quad du = 36\,dx$

$x = \frac{1}{12} \rightarrow u = 1 + 3 = 4$

$x = \frac{2}{9} \rightarrow u = 1 + 8 = 9$

$$= \int_4^9 \frac{1}{36} u^{1/2}\, du.$$

$$= \frac{1}{36} \cdot \frac{2}{3} u^{3/2} \Big|_4^9$$

$$= \frac{1}{54}(27 - 8)$$

$$= \boxed{\frac{19}{54}}$$