MATH 121

ANALYSIS OF FUNCTIONS

GOALS

- ☐ BE ABLE TO DESCRIBE THE RELATIONSHIPS BETWEEN f, f', AND f''
- ☐ DETERMINE THE INTERVALS ON WHICH f IS INCREASING, DECREASING, CONCAVE UP, AND CONCAVE DOWN.
- ☐ FIND RELATIVE EXTREMA FOR A FUNCTION f
- ☐ FIND INFLECTION POINTS FOR A FUNCTION f
- ☐ FIND ABSOLUTE EXTREMA FOR A FUNCTION f

CENTRAL QUESTION: WHAT DOES f' TELL US ABOUT f? AND f''?

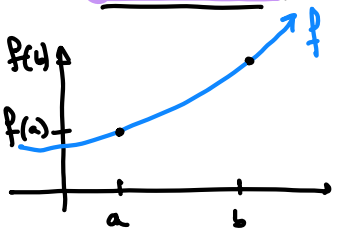
RECALL:

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \qquad (b > a)$$

($b = a + h$)

($b - a$ ALWAYS POSITIVE.)

INCREASING:

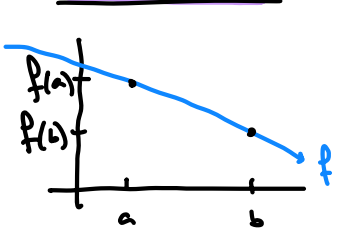


$f(b) > f(a)$

$f(b) - f(a) > 0$

so $\boxed{f'(a) > 0}$

DECREASING:



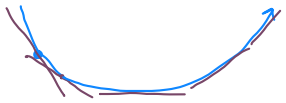
$f(a) > f(b)$

$f(b) - f(a) < 0$

so $\boxed{f'(a) < 0}$

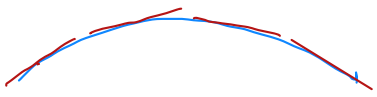
How about f''?

$f'' > 0 \longrightarrow f'$ is increasing $\longrightarrow f$ concave up.



slopes are increasing (tangents rotate counterclockwise).

$f'' < 0 \longrightarrow f'$ is decreasing $\longrightarrow f$ concave down.



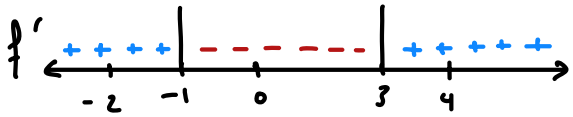
Ex: $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 1$

- Find intervals where f is increasing and decreasing.

  - Find f': $f'(x) = x^2 - 2x - 3$
  - Identify points where f' can change sign
    1. $f' = 0$: $x^2 - 2x - 3 = 0$

       $(x-3)(x+1) = 0$

       $\boxed{x = 3, -1}$
    2. f' undefined: none.
  - Make sign chart: choose a "test point" in each interval.



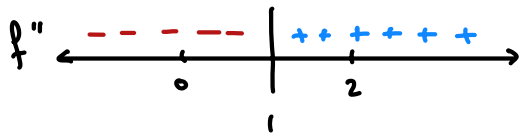
$f(-2) = (-5)(-1) = +$

  - Conclusions: f is increasing on $(-\infty, -1)$, $(3, \infty)$

    decreasing on $(-1, 3)$

· FIND INTERVALS WHERE f IS CONCAVE UP AND CONCAVE DOWN.

- FIND f'': $f''(x) = 2x - 2$
- FIND POINTS WHERE f'' CAN CHANGE SIGN:
  1. $f'' = 0$: $2x - 2 = 0 \longrightarrow \boxed{x = 1}$
  2. f'' UNDEF: NONE.
- SIGN CHART:



- CONCLUSIONS: f IS CONCAVE UP ON $(1, \infty)$
  CONCAVE DOWN ON $(-\infty, 1)$

EX: $f(x) = 2x + 3x^{2/3}$

· INC/DEC:

- FIND $f'(x)$: $f'(x) = 2 + 2x^{-1/3}$
- f' CAN CHANGE SIGN:
  1. $f' = 0$: $2 + 2x^{-1/3} = 0 \longrightarrow 2x^{-1/3} = -2$

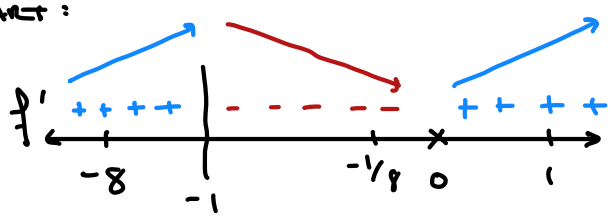
$$\left(x^{-1/3}\right)^{-3} = (-1)^{-3}$$

$$\boxed{x = -1}$$

2. f' UNDEF: $f'(x) = 2 + \frac{2}{\sqrt[3]{x}}$ IS UNDEF AT $\boxed{x = 0}$

NOTICE: f IS DEF. AT $x = 0$.

- SIGN CHART:



$$f'(-1/8) = 2 + \frac{2}{\sqrt[3]{-\frac{1}{8}}}$$
$$= 2 + \frac{2}{-\frac{1}{2}}$$
$$= 2 - 4$$

$$f'(-8) = 2 + \frac{2}{\sqrt[3]{-8}}$$
$$= 2 + \frac{2}{-2}$$

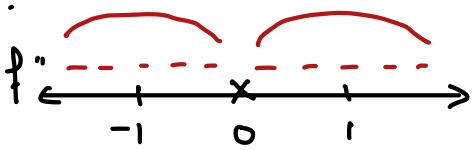
· CONCAVITY:

- FIND f'': $f''(x) = -\frac{2}{3}x^{-4/3} = \frac{-2}{3\sqrt[3]{x^4}}$

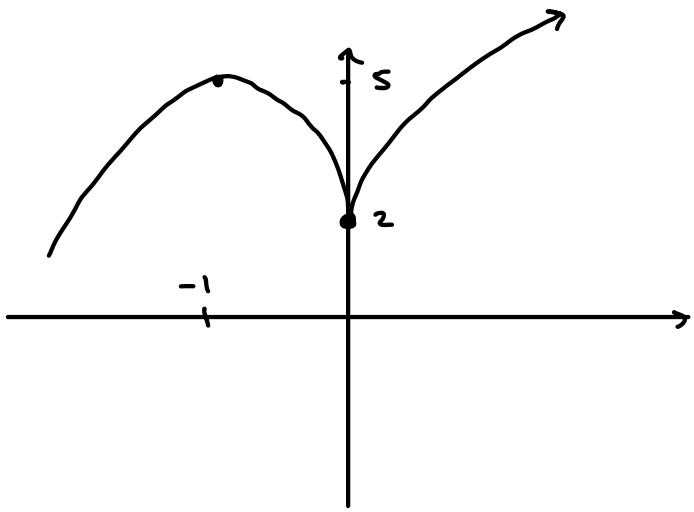
- f'' CHANGES SIGN:

  1. $f'' = 0$: NONE
  2. f'' UNDEF : $x = 0$

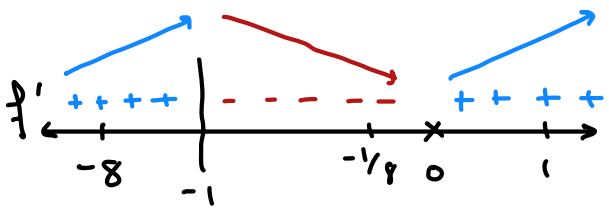
- SIGN CHART:

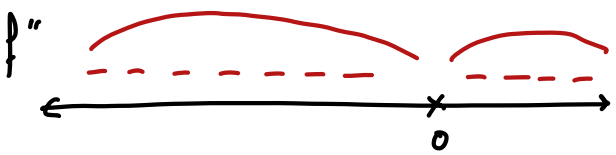


$f(x) = 2 + 3x^{2/3}$



$f(-1) = 5$ $f(0) = 2$





f IS...

INC: $(-\infty, -1)$ $(0, \infty)$

DEC: $(-1, 0)$

CONC. UP: N/A

CONC DOWN: $(-\infty, 0)$ $(0, \infty)$

WE ALSO CAN SEE THAT f HAS A

RELATIVE MAX AT $x = -1$

RELATIVE MIN AT $x = 0$

RECAP

$f' > 0 \longrightarrow f$ INCREASING

$f' < 0 \longrightarrow f$ DECREASING

$f'' > 0 \longrightarrow f'$ INC. $\longrightarrow f$ CONC. UP

$f'' < 0 \longrightarrow f$ DEC. $\longrightarrow f$ CONCAVE DOWN

PROCESS:

1. FIND f'
2. FIND CRIT. PTS:
   i) $f' = 0$ ii) f' UNDEF.
3. USE (2) TO MAKE SIGN CHART
4. READ CONCLUSIONS FROM (3).

EXTREMA:

- f HAS REL. MAX IF IT GOES FROM INC. TO DEC. (f' GOES + TO −)
- f HAS REL. MIN IF IT GOES FROM DEC. TO INF (f' GOES − TO +)
- f HAS AN INFLECTION POINT IF IT CHANGES CONCAVITY. (f'' CHANGES SIGN)

Ex: $f(x) = xe^x$

① $f'(x) = e^x + xe^x$

② $e^x + xe^x = 0$

$e^x(1 + x) = 0$

$e^x = 0$ NO SOL'N

$1 + x = 0$ $\boxed{x = -1}$

③ f': − − − − − | + + + + +

-2 -1 0

USE FACTORED VERSION

$f'(-2) = e^{-2}(1 + (-2))$

$(+)\ (-)$

f INC ON $(-1, \infty)$ AND DEC. $(-\infty, -1)$

f HAS A RELATIVE MIN AT $x = -1$

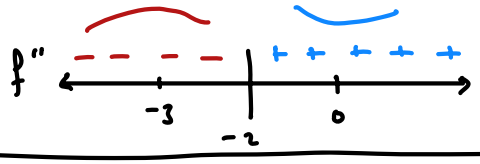
$f''(x) = e^x + e^x + xe^x$

$= 2e^x + xe^x$

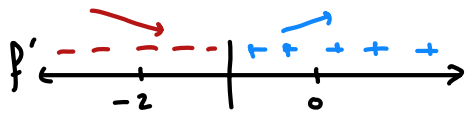
$$2e^x + xe^x = 0$$

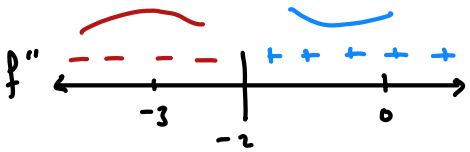
$$e^x(2+x) = 0$$

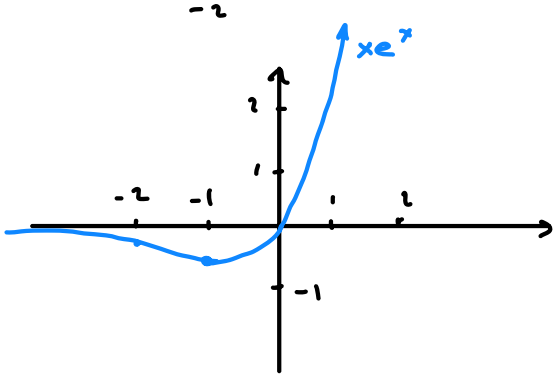
$e^x = 0$ NONE $2 + x = 0$ $\boxed{x = -2}$



f IS CONCAVE UP ON $(-2, \infty)$ AND DOWN ON $(-\infty, -2)$ w/ AN INFLECTION PT. AT $\boxed{x = -2}$







$f(-1) = -1e^{-1} = -\frac{1}{e}$

END BEHAVIOR:

$$\lim_{x \to \infty} xe^x = \infty$$

$$\lim_{x \to -\infty} xe^x \longrightarrow \infty \cdot 0 \quad \text{NEED L'HÔP}$$

$$= \lim_{x \to -\infty} \frac{x}{e^{-x}} \quad \frac{x \to \infty}{e^{-x} \to \infty}$$

$$\overset{LH}{=} \lim_{x \to -\infty} \frac{1}{-e^{-x}} \quad \frac{1 \to 1}{-e^{-x} \to \infty}$$

$$= \boxed{0}$$

TRY:

$$f(x) = \frac{2x}{x^2+1}$$

$$f'(x) = \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2}$$

$$= \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2}$$

$$= \boxed{\frac{2 - 2x^2}{(x^2+1)^2}}$$

$$f''(x) = \frac{(x^2+1)^2(-4x) - 2(x^2+1)(2x)(2-2x^2)}{(x^2+1)^4}$$

$$= \frac{(x^2+1)(-4x) - 4x(2-2x^2)}{(x^2+1)^3}$$

$$= \frac{-4x^3 - 4x - 8x + 8x^2}{(x^2+1)^3}$$

$$f''(x) = \boxed{\frac{4x^3 - 12x}{(x^2+1)^3}}$$

$$f'(x) = \frac{2-2x^2}{(x^2+1)^2}$$

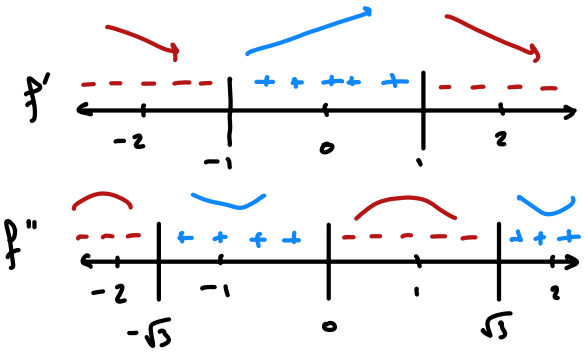
$$f''(x) = \frac{4x^3 - 12x}{(x^2+1)^3}$$

$2 - 2x^2 = 0$

$\boxed{x = \pm 1}$

$4x^3 - 12x = 0$

$\boxed{x = \pm\sqrt{3}, 0}$



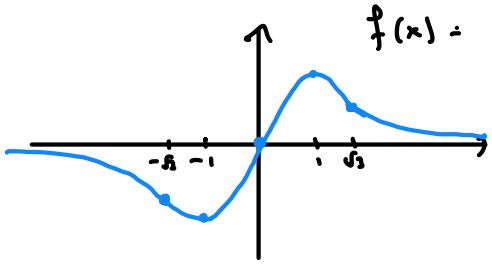
$4x^3 - 12x = 4x(x^2 - 3)$

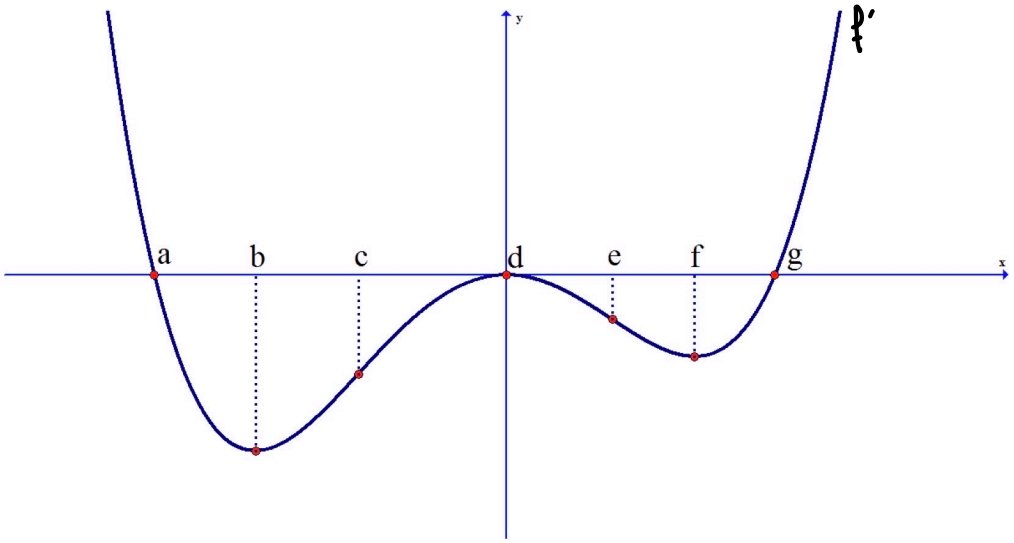
f IS INC ON $(-1,1)$ AND DEC. $(-\infty,-1)$ AND $(1,\infty)$

CONC. UP $(-\sqrt{3},0)$ AND $(\sqrt{3},\infty)$ AND CONC. DOWN $(-\infty,-\sqrt{3})$ AND $(0,\sqrt{3})$

f HAS A REL. MAX AT $x=1$ AND REL. MIN AT $x=-1$ AND INF. PTS AT $x = -\sqrt{3}, 0, \sqrt{3}$

$f(x) = \frac{2x}{x^2+1}$





f IS INC: $(-\infty, a)$ (g, ∞)

f IS DEC: (a, d) (d, g)

REL. MAX: $x = a$

REL MIN: $x = g$.

CONC. UP: (b, d)
(f, ∞)

CONC DOWN: $(-\infty, b)$
(d, f)

TRY:

$$f(x) = x^4 - 2x^2$$

FIND WHERE f IS INCREASING/DECREASING, CONCAVE UP/DOWN, HAS RELATIVE EXTREMA, AND HAS INFLECTION POINTS.

$$f'(x) = 4x^3 - 4x$$

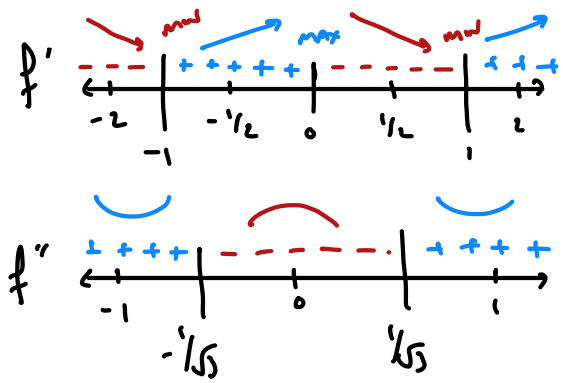
$$4x(x^2 - 1) = 0$$

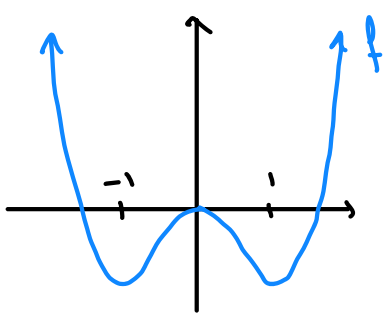
$$\boxed{x = 0, \pm 1}$$

$$f''(x) = 12x^2 - 4$$

$$4(3x^2 - 1) = 0$$

$$\boxed{x = \pm \frac{1}{\sqrt{3}}}$$





INC: $(-1, 0)$ $(1, \infty)$

DEC: $(-\infty, -1)$ $(0, 1)$

UP: $(-\infty, -1/\sqrt{3})$
$(1/\sqrt{3}, \infty)$

DOWN: $(-1/\sqrt{3}, 1/\sqrt{3})$

MAX: $x = 0$

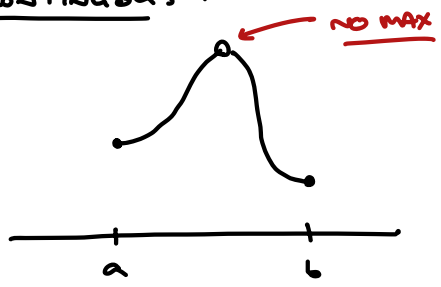
MIN: $x = \pm 1$

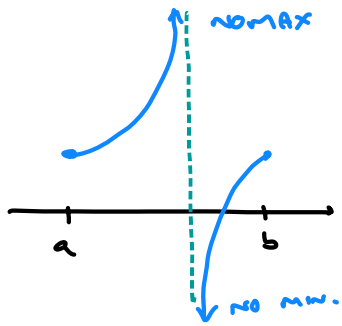
INF. PTS: $x = \pm \frac{1}{\sqrt{3}}$

ABSOLUTE EXTREMA:

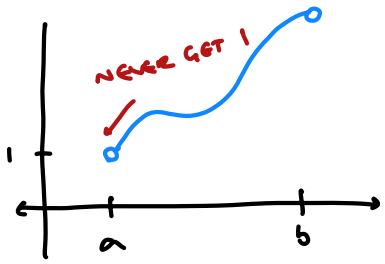
EXTREME VALUE THEOREM: IF f IS CONTINUOUS ON $[a,b]$, THEN IT MUST HAVE AN ABSOLUTE MAX AND AN ABSOLUTE MIN ON $[a,b]$ AT EITHER A CRITICAL POINT OR AN END POINT.

- WHY CONTINUOUS?

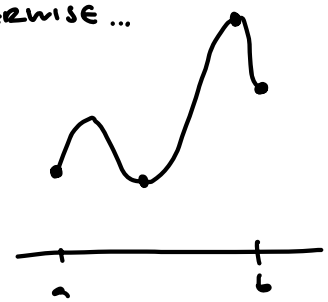


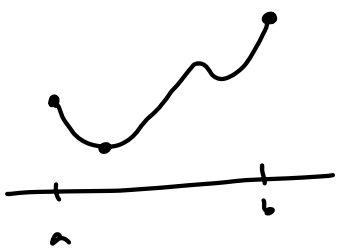


- WHY CLOSED INTERVAL $[a,b]$?



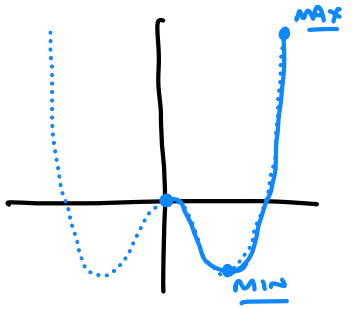
- OTHERWISE ...







Ex: $f(x) = x^4 - 2x^2 \; ; \; [0, 2]$



- GET $f'(x)$: $f'(x) = 4x^3 - 4x$
- FIND CRITICAL PTS IN $[0,2]$

$$4x(x^2 - 1) = 0$$

$x = -1, 0, 1$ (NOT INSIDE)

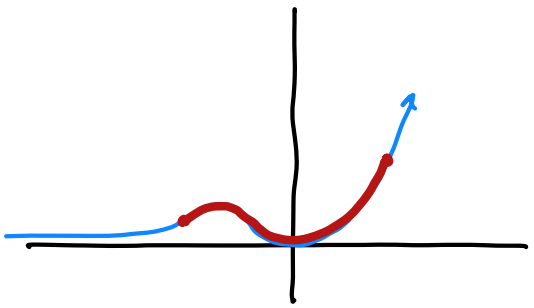
- EVALUATE f AT CRIT. PTS. AND END PTS:

$f(0) = 0$

$f(1) = \boxed{-1}$ ABS. MIN

$f(2) = 16 - 2 \cdot 4 = \boxed{8}$ ABS. MAX

Ex: $g(x) = x^2 e^x \; ; \; [-3, 1]$



$$g'(x) = 2xe^x + x^2 e^x$$

$$xe^x(2 + x) = 0$$

$x = 0, -2$

COMPARE:

$g(-3) = 9e^{-3} = \frac{9}{e^3} \approx \frac{9}{27}$

$g(-2) = 4e^{-2} = \frac{4}{e^2} \approx \frac{4}{9}$

$g(0) = 0$ MIN

$g(1) = e$ MAX

ABSOLUTE EXTREMA RECAP:

- FOR CONTINUOUS f ON $[a,b]$:
  - FIND f'
  - FIND ALL CRIT. PTS. IN $[a,b]$
  - COMPARE f-VALUES AT CRIT. PTS. AND END PTS.
- FOR CONTINUOUS f ON (a,b) OR $(-\infty,\infty)$ OR (a,∞) OR $(-\infty,b)$
  - FIND f'
  - FIND ANY CRIT. PTS IN INTERVAL
  - COMPARE f-VALUES AT CRIT. PTS AND LIMITS AND END PTS.

Ex: $f(x) = x^3 - 3x$; $[-1,2]$

$f'(x) = 3x^2 - 3$

$3x^2 - 3 = 0$

$3(x^2-1) = 0 \longrightarrow$ CRIT PTS: $x = \pm 1$

COMPARE:

$f(-1) = (-1)^3 - 3(-1) = 2$

$f(1) = 1 - 3 = -2$

$f(2) = 2^3 - 3(2) = 2$

ABS MAX: 2

ABS. MIN: -2

WHAT IF WE HAD $f(x) = x^3 - 3x$ ON $(-1,2)$?

STILL FIND f' AND CRIT PTS. $x = \cancel{-1}, 1$

COMPARE:

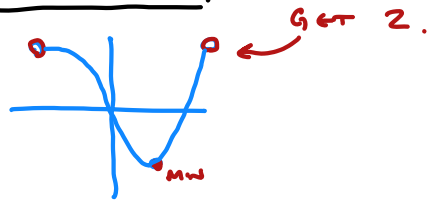
$\lim_{x \to -1} (x^3 - 3x) = 2$

$f(1) = -2$

$\lim_{x \to 2} (x^3 - 3x) = 2$

ABS. MIN: -2

NO ABS MAX ← WE NEVER GET 2.



Ex: $f(x) = \dfrac{1}{x^2+1}$; $(-\infty, \infty)$ $\quad (x^2+1)^{-1}$

$$f'(x) = -(x^2+1)^{-2}(2x)$$

$$= \frac{-2x}{(x^2+1)^2}$$

CRIT. PTS: $-2x = 0$

$\boxed{x = 0}$

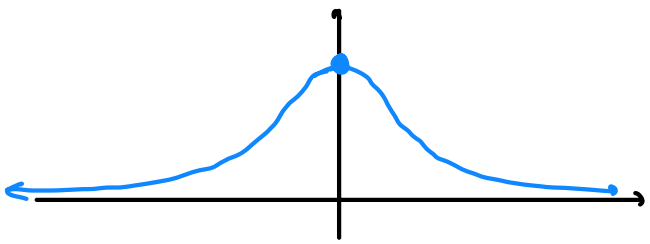
COMPARE:

$$\lim_{x \to -\infty} \frac{1}{x^2+1} = \boxed{0}$$

$$f(0) = \frac{1}{1} = \boxed{1}$$

$$\lim_{x \to +\infty} \frac{1}{x^2+1} = \boxed{0}$$

$\boxed{\text{ABS. MAX: 1}}$ $\quad \boxed{\text{NO ABS MIN}}$



Ex: $f(x) = xe^{-x^2}$ $\quad (-\infty, \infty)$

$$f'(x) = e^{-x^2} + x(-2x)e^{-x^2}$$

$$= e^{-x^2}(1-2x^2)$$

$e^{-x^2} = 0$ NO SOL'N

$1 - 2x^2 = 0 \longrightarrow x = \pm \frac{1}{\sqrt{2}}$

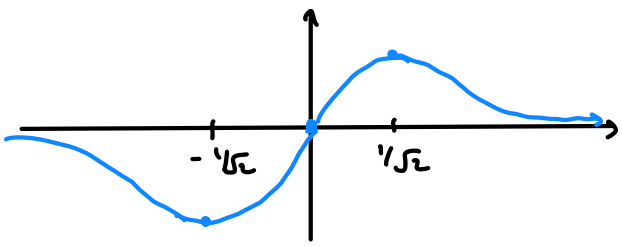
$$\lim_{x \to -\infty} xe^{-x^2} \longrightarrow -\infty \cdot 0$$

$$= \lim \frac{x}{e^{x^2}}$$

$$\overset{L'H}{=} \lim \frac{1}{2xe^{x^2}} \longrightarrow \frac{1}{\infty}$$

$$= \boxed{0}$$

$$\lim_{x \to \infty} xe^{-x^2} = \boxed{0}$$



$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} e^{-1/2} = \frac{1}{\sqrt{2e}}$$

$$f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} e^{-1/2} = -\frac{1}{\sqrt{2e}}$$

$\boxed{\text{ABS MAX: } \frac{1}{\sqrt{2e}} \quad \text{ABS MIN: } -\frac{1}{\sqrt{2e}}}$