MATH 121

AN INTUITIVE APPROACH TO LIMITS

GOALS

- ☐ WRITE A CLEAR DEFINITION OF THE LIMIT OF A FUNCTION
- ☐ DEFINE ONE-SIDED LIMITS
  - ☐ USE ONE-SIDED LIMITS TO JUSTIFY WHETHER A LIMIT EXISTS OR NOT
- ☐ GIVEN THE GRAPH OF A FUNCTION, DETERMINE THE LIMIT OF f AS x APPROACHES A FINITE VALUE (FOR BOTH ONE- AND TWO-SIDED LIMITS)
- ☐ DEFINE VERTICAL ASYMPTOTES IN TERMS OF LIMITS
  - ☐ DEFINE $+\infty$, $-\infty$

A MOTIVATING QUESTION

Q: IS $0.\overline{9} = 1$? (OR IS IT "ALMOST" 1?)

RECALL: $0.\overline{9}$ MEANS $0.999\ldots$, $0.\overline{12}$ MEANS $0.121212\ldots$

WE HAVE TO BE CLEARER ON WHAT WE MEAN HERE.

DEFINE A FUNCTION

$$f(1) = 0.9$$
$$f(2) = 0.99$$
$$f(3) = 0.999$$
$$\vdots$$
$$f(n) = 0.\underbrace{999\ldots.9}_{n\ 9\text{'s}}$$

WE CAN MAKE THE VALUE OF f AS CLOSE TO 1 AS WE LIKE

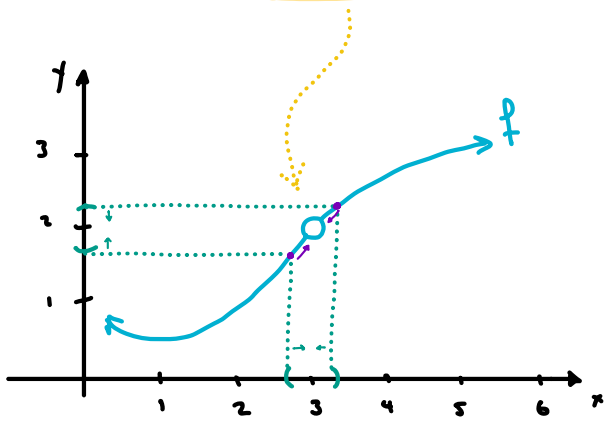
BY MAKING n LARGE ENOUGH

WE WRITE $\lim_{n\to\infty} f(n) = 1$ ← THIS IS A LIMIT

A: $0.\overline{9} = \lim_{n\to\infty} f(n) = 1$

TRANSLATING THIS DEFINITION TO GRAPHS

WHAT'S GOING ON AT $x = 3$ ON THIS GRAPH?

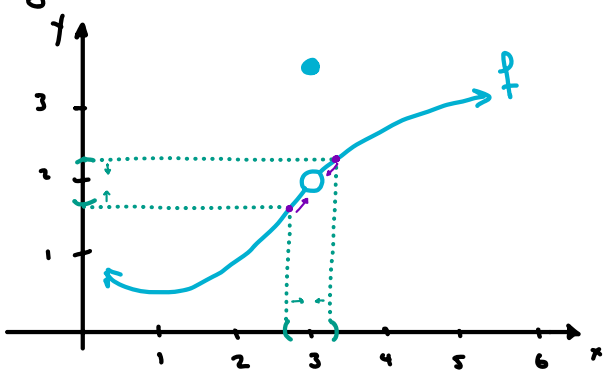


- $f(3)$ IS UNDEFINED ← (3,2) ISN'T INCLUDED

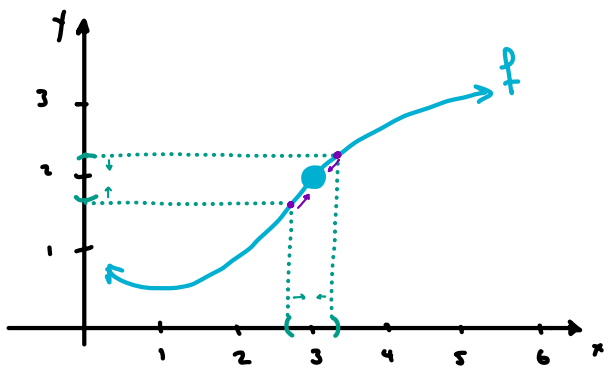
* $\lim_{x \to 3} f(x) = 2$ B/C WE CAN MAKE f AS CLOSE TO 2 AS WE LIKE BY MAKING x CLOSE ENOUGH TO 3.

NOTE: WHAT $f(a)$ IS EQUAL TO AND WHAT $\lim_{x \to a} f(x)$ IS ARE <u>SEPARATE</u> QUESTIONS.

e.g.



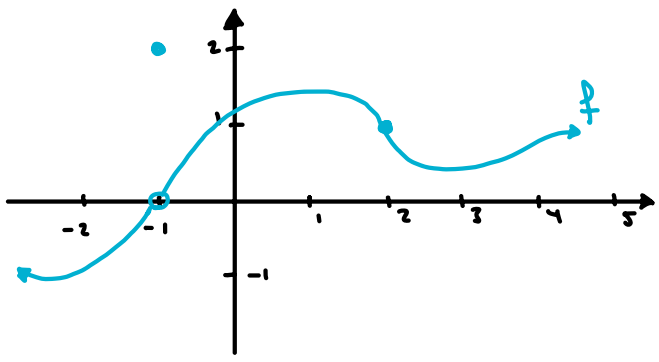
$f(3) = 3.5$ BUT $\lim_{x \to 3} f(x) = 2$



$f(3) = 2$ AND $\lim_{x \to 3} f(x) = 2$

WHEN $\lim_{x \to a} f(x) = f(a)$, WE SAY f IS <u>CONTINUOUS</u> AT $x = a$.

EX DETERMINE $f(-1)$, $f(2)$, $\lim_{x \to -1} f(x)$, AND $\lim_{x \to 2} f(x)$ USING THE GRAPH



$f(-1) = 2$

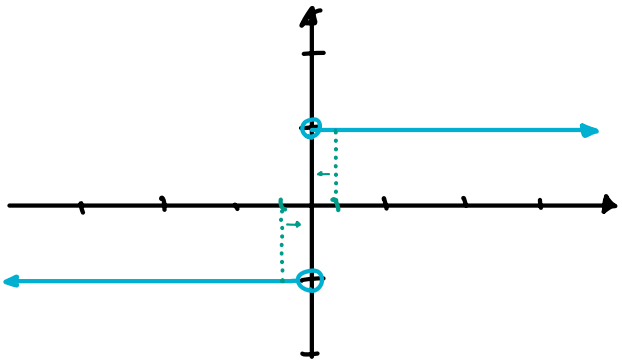
$f(2) = 1$

$\lim_{x \to -1} f(x) = 0$

$\lim_{x \to 2} f(x) = 1$

AN ODD EXAMPLE

SKETCH THE GRAPH OF $f(x) = \dfrac{|x|}{x}$



WHAT'S GOING ON AT $x = 0$?

$\lim_{x \to 0} f(x)$ DOES NOT EXIST (DNE)

WE CAN BE MORE SPECIFIC

ONE-SIDED LIMITS

DEFINITION

(LEFT) $\lim_{x \to a^-} f(x) = L$: WE CAN MAKE f AS CLOSE TO L AS WE LIKE BY MAKING x CLOSE ENOUGH TO a FOR $x < a$

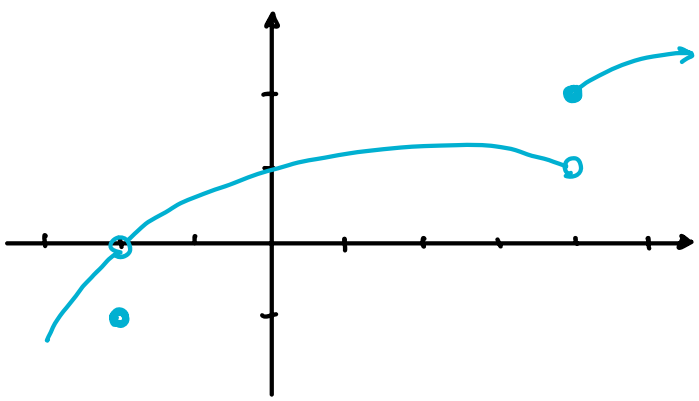
(RIGHT) $\lim_{x \to a^+} f(x) = R$: WE CAN MAKE f AS CLOSE TO R AS WE LIKE BY MAKING x CLOSE ENOUGH TO a FOR $x > a$

e.g. $\lim_{x \to 0^-} \dfrac{|x|}{x} = -1$ WHILE $\lim_{x \to 0^+} \dfrac{|x|}{x} = 1$

IMPORTANT!

$\lim_{x \to a} f(x) = L$ IF AND ONLY IF $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

EX DETERMINE $f(-2)$, $\lim_{x \to -2} f(x)$, $f(4)$, $\lim_{x \to 4} f(x)$. IF A LIMIT DNE, EXPLAIN.



$f(-2) = 0$

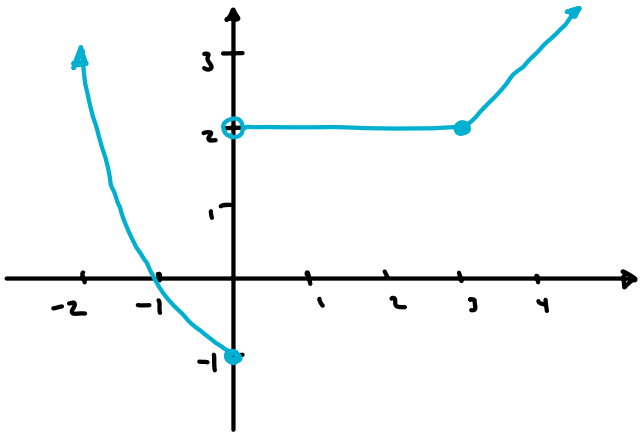
$\lim_{x \to -2} f(x) = -1$

$f(4) = 2$

$\lim_{x \to 4} f(x)$ DNE B/C $\lim_{x \to 4^-} f(x) = 1 \neq \lim_{x \to 4^+} f(x) = 2$

EX SKETCH THE FOLLOWING PIECEWISE FUNCTION AND DETERMINE $f(0)$, $f(3)$, $\lim_{x\to 0} f(x)$, AND $\lim_{x\to 3} f(x)$.

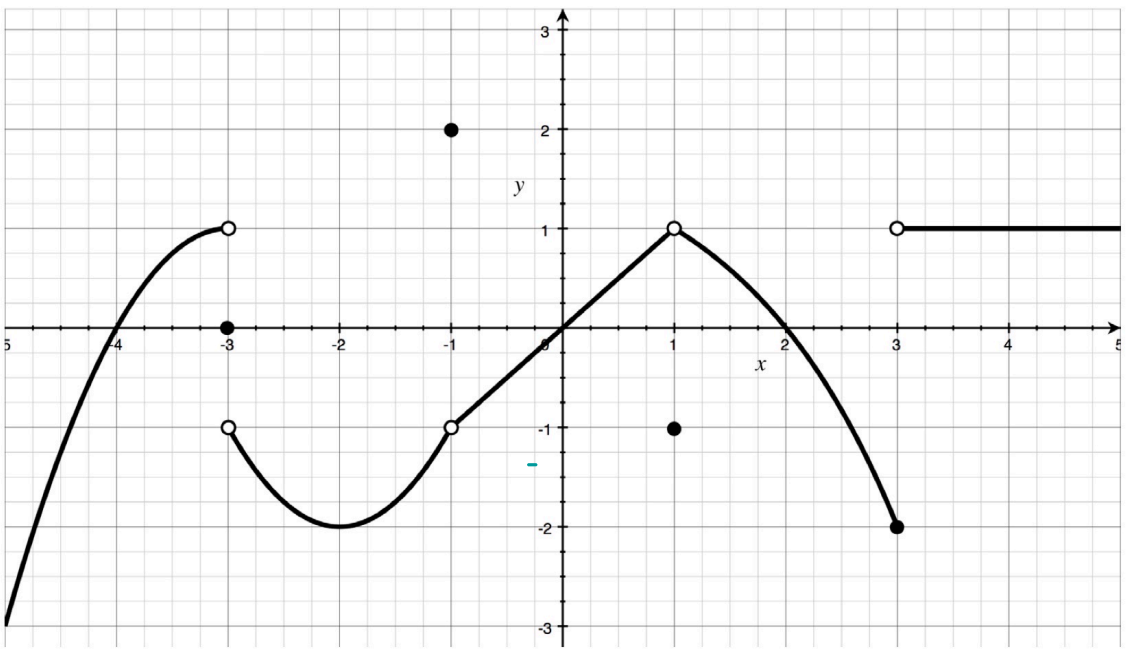
$$f(x) = \begin{cases} x^2 - 1, & x \le 0 \\ 2, & 0 < x \le 3 \\ x - 1, & x > 3 \end{cases}$$



$f(0) = -1$, $\lim_{x\to 0} f(x)$ ONE; $f(3) = 2$; $\lim_{x\to 3} f(x) = 2$

$\lim_{x\to 0^-} f(x) = -1 \neq \lim_{x\to 0^+} f(x) = 2$

EX USE THE GRAPH TO DETERMINE THE ONE- AND TWO-SIDED LIMITS (IF THEY EXIST) AS WELL AS THE FUNCTION VALUES AT $x = -3, -1, 1, 3$



$\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1} f(x) = 1$

$f(1) = -1$

$\lim_{x\to 3^-} f(x) = -2$

$\lim_{x\to 3^+} f(x) = 1$

$\lim_{x\to 3} f(x)$ DNE

$f(3) = -2$

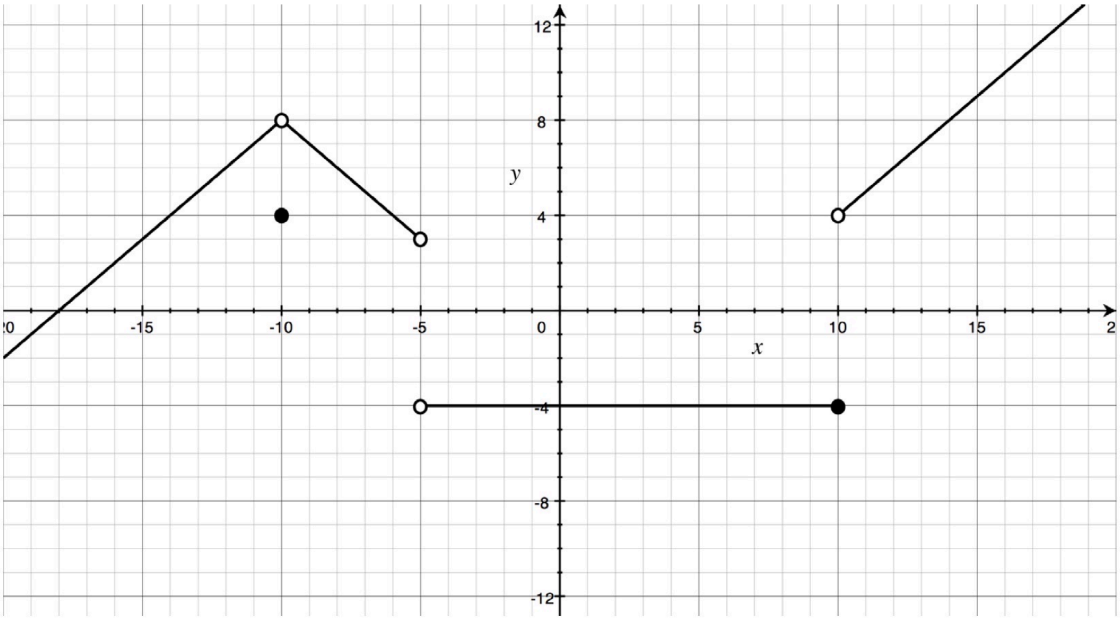
$\lim_{x\to -3^-} f(x) = 1$ $\quad \lim_{x\to -3} f(x)$ DNE

$\lim_{x\to -3^+} f(x) = -1$ $\quad f(-3) = 0$

$\lim_{x\to -1^-} f(x) = -1$ $\quad \lim_{x\to -1} f(x) = -1$

$\lim_{x\to -1^+} f(x) = -1$ $\quad f(-1) = 2$

EX USE THE GRAPH TO DETERMINE THE LIMITS AND FUNCTION VALUES AT $x = -10, -5, 10$ (IF A TWO-SIDE LIMIT DOESN'T EXIST, WRITE THE ONE-SIDED LIMITS)



$\lim_{x \to -10} f(x) = 8$

$f(-10) = 4$

$\lim_{x \to -5} f(x)$ DNE $\quad \lim_{x \to -5^+} f(x) = -4$

$\lim_{x \to -5^-} f(x) = 3 \quad f(-5)$ IS UNDEF

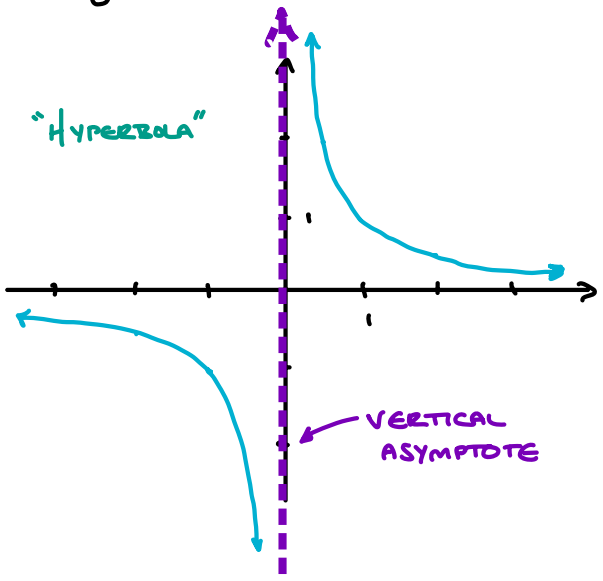
$\lim_{x \to 10} f(x)$ DNE $\quad \lim_{x \to 10^+} f(x) = 4$

$\lim_{x \to 10} f(x) = -4 \quad f(10) = -4$

VERTICAL ASYMPTOTES

- **VISUAL AIDS** WE INCLUDE WITH A GRAPH TO **HIGHLIGHT CERTAIN BEHAVIORS.** (NOT A PART OF GRAPH)
- VERT. ASYMPTOTES TELL US THAT **$f(x)$ GROWS OR DECREASES WITHOUT BOUND** AS x APPROACHES a FROM ONE SIDE OR BOTH.

e.g. $f(x) = \frac{1}{x}$



x	2	1	$1/2$	$1/10$	$1/1000$	$1/100000$
$f(x)$	$1/2$	1	2	10	1000	100000

AS $x \longrightarrow 0^+$, $f(x) \longrightarrow \infty$

$f(x)$ GROWS WITHOUT BOUND

SIMILARLY AS $x \longrightarrow 0^-$, $f(x) \longrightarrow -\infty$

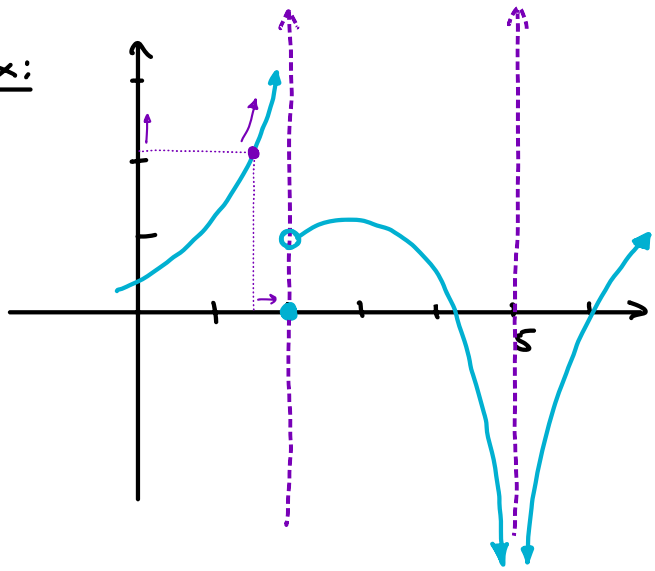
WE WRITE...

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

IN GENERAL, WE HAVE AN VERTICAL ASYMPTOTE AT $x = a$ IF

EITHER $\lim_{x \to a^-} f = \pm\infty$ OR $\lim_{x \to a^+} f = \pm\infty$.

EX:



$\lim_{x \to 2^-} f(x) = +\infty$

$\lim_{x \to 2^+} f(x) = 1$

$\Big\}$ $\lim_{x \to 2} f(x)$ DNE

$f(2) = 0$

$\lim_{x \to 5^-} f(x) = -\infty$

$\lim_{x \to 5^+} f(x) = -\infty$

$\Big\}$ $\lim_{x \to 5} f(x) = -\infty$

$f(5)$ IS UNDEF.